TOWARDS PUBLIC MARKET EQUIVALENCE

Public benchmarking for private equity

Christian Tausch

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AssetMetrix

Agenda

1. Introduction
2. Public Market Equivalent (PME) Approaches
3. Stochastic Discount Factor (SDF) for Private Equity
4. Data & Model Estimation
5. Conclusion

1 INTRODUCTION

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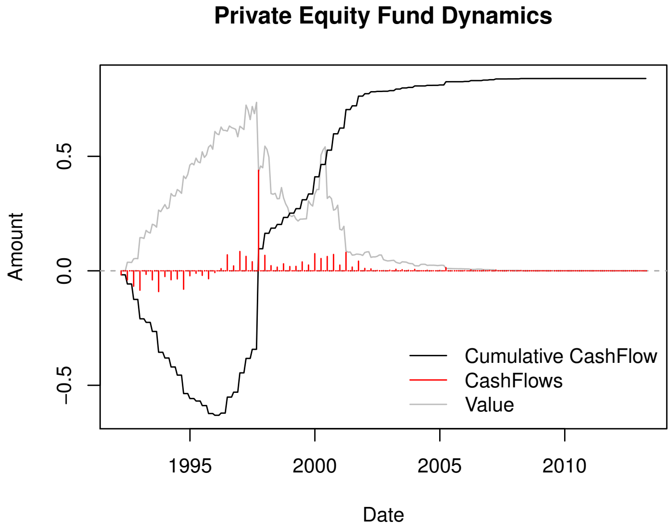
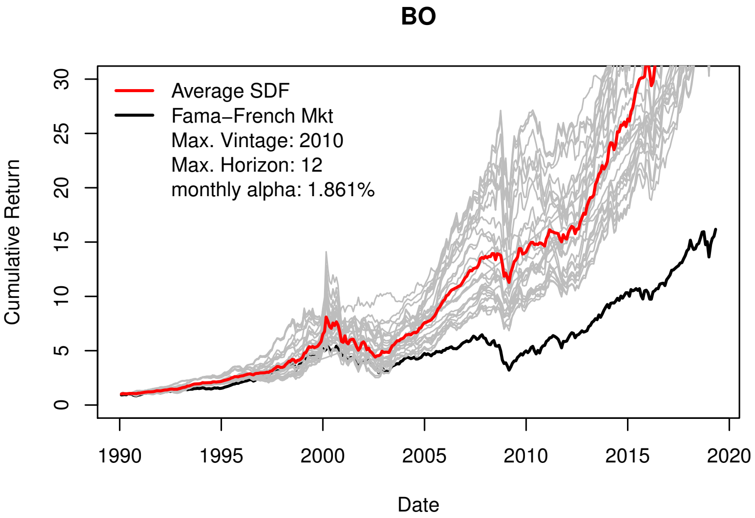
Title	Towards Public Market Equivalence
Status	Working paper (being planned)
Idea	Compare and unify common Public Market Equivalent (PME) approaches from a (1) Stochastic Discount Factor (SDF) and (2) cash flow replication perspective.
Aim	Quantify public market out/under-performance of private equity fund investments within a comprehensible and rigorous framework.
Application	Create more tailored benchmarks.


Private Equity Fund Dynamics
Amount
0.5
0.0
−0.5
1995
2000
2005
2010
Date
Cumulative CashFlow
CashFlows
Value

Public Market Equivalent (PME)

- Several competing methodologies to make private equity fund performance comparable to public equity.
- Benchmark private to public equity to determine ex-post best investment alternative.
- Challenges for comparison:
    - Closed-end private equity fund structure
    - No tradable market values for PE funds –> no return time-series like in public equity (stale pricing)
    - Observed fund cash flows are only source of reliable hard data

How to convert a panel of cash flows into a total return index?



Private equity fund i is characterized for discrete times t by:

Net Asset Value $V_{i,t}$ (fund value proxy)

Contribution $C_{i,t}$ (fund inflow from investors)

Distribution $D_{i,t}$ (fund outflow to investors)

Public market is given by:

Asset $S_{j,t} \geq 0$ (price of non dividend paying asset)

SDF $\Psi_{0,t} > 0$ (stochastic discount factor from t to 0)

Numeraire $S_t^* > 0$ (default-free asset to serve as SDF)

Predictor $Z_{k,t}$ (macro indicator)

2 PME APPROACHES

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2.1 PME: PRICING VS. REPLICATION

Pricing of observed fund cash flows:

- [Kaplan and Schoar, 2005]: discount all observed cash flows by numeraire S^*.
- [Korteweg and Nagel, 2016, Driessen et al., 2012]: discount all observed cash flows by more general SDF Ψ.

Replication of observed fund cash flows:

- [Long and Nickels, 1996]: match all in- and outflows, invest/finance residual value in/by public index.
- PME+: match all inflows and invest them in public market index, scale PME outflows by ex-post constant
- modified PME: match all inflows and invest them in public market index, scale PME outflows proportional to observed fund distribution ratio.
- Tausch (2019): base replication strategy on observed fund valuation (not observed cash flows).

Price of fund i: discount all observed cash flows by SDF $\Psi > 0$:

$$P_{0,T_i} = \sum_{\tau=1}^{T_i} (D_\tau - C_\tau) \cdot \Psi_{0,\tau}$$

with fund liquidation date T_i.

Price of **replication** strategy for fund i: discount all replication cash flows by same SDF $\Psi > 0$:

$$R_{0,T_i} = \sum_{\tau=1}^{T_i} (A_\tau - B_\tau - \lambda_\tau) \cdot \Psi_{0,\tau}$$

with divestment $A \geq 0$, investment $B \geq 0$, cost $\lambda \geq 0$.

Discount observed cash flows by numeraire portfolio [Long and Nickels, 1996], [Kaplan and Schoar, 2005], [Long, 2008]:

$$P_{0,t}^{(LN96)} = P_{0,t}^{(KS05)} = \sum_{\tau=1}^{t} (D_\tau - C_\tau) \cdot \frac{S_0^*}{S_\tau^*}$$

Discount observed cash flows by linear SDF [Driessen et al., 2012]:

$$P_{0,t}^{(DLP12)} = \sum_{\tau=1}^{t} (D_\tau - C_\tau) \cdot \prod_{h=0}^{\tau} \left[\frac{S_h^{(rf)}}{S_{h-1}^{(rf)}} + \alpha + \beta \left(\frac{S_h^{(market)}}{S_{h-1}^{(market)}} - \frac{S_h^{(rf)}}{S_{h-1}^{(rf)}} \right) \right]$$

Discount observed cash flows by exponential affine SDF (with m_d linear function of market factors) [Korteweg and Nagel, 2016]:

$$P_{0,t}^{(KN16)} = \sum_{\tau=1}^{t} (D_\tau - C_\tau) \cdot \exp \left(- \sum_{d=0}^{\tau} m_d \right)$$

2.4 REPLICATION: PME+ AND MODIFIED PME

PME+ and modified PME both use observed fund contributions and invest them into public market. Replicated distributions need scaling.

PME+ unpredictable strategy, since Γ_T just known ex-post at time T:

$$R_{0,t}^{(PME+)} = \sum_{\tau=1}^{t} [\Gamma_T \cdot D_\tau - C_\tau] \cdot \Psi_{0,\tau} \tag{1}$$

$$\Gamma_T = \frac{V_T + \sum_{\tau=1}^{T} C_\tau \frac{S_T}{S_\tau}}{\sum_{\tau=1}^{T} D_\tau \frac{S_T}{S_\tau}} \tag{2}$$

Modified PME predictable strategy with τ-information:

$$R_{0,t}^{(mPME)} = \sum_{\tau=1}^{t} [\frac{D_\tau}{D_\tau + V_\tau} \cdot (\mathring{V}_{\tau-1} \cdot \frac{S_\tau}{S_{\tau-1}} + C_\tau) - C_\tau] \cdot \Psi_{0,\tau} \tag{3}$$

$$\mathring{V}_\tau = (1 - \frac{D_\tau}{D_\tau + V_\tau}) \cdot (\mathring{V}_{\tau-1} \cdot \frac{S_\tau}{S_{\tau-1}} + C_\tau) \tag{4}$$

2.5 REPLICATION: VALUE-BASED HEDGING APPROACH

Value-based discounted replication cash flow (Tausch, 2019):

$$R_{0,t}^{(T19)} = \sum_{\tau=1}^{t} \beta \cdot Z_{\tau-1} \cdot V_{\tau-1} \cdot \left(\frac{S_{\tau}^{(+)}}{S_{\tau-1}^{(+)}} - \frac{S_{\tau}^{(-)}}{S_{\tau-1}^{(-)}} - \lambda \right) \cdot \frac{S_0^*}{S_\tau^*}$$

- In contrast to the other gain processes no knowledge of C_τ and D_τ is required –> just V_τ.
- How to bound possible losses associated with this strategy?
- Further research: Possible to just use average fund information (typified pattern)? This means methodology that requires no information on actual C_τ, D_τ, and V_τ.

Which PME approaches can be used for pricing/replication?

	Variables needed			Suitable for	
Approach	C_τ	D_τ	V_τ	Pricing	Replication
LN96	yes	yes	no	yes	no
KS05	yes	yes	no	yes	no
DLP12	yes	yes	no	yes	no
KN16	yes	yes	no	yes	no
PME+	yes	yes	no	no	no
mPME	yes	yes	yes	no	yes
T19	no	no	yes	no	yes

3 SDF FOR PRIVATE EQUITY

- SDF = arbitrage free model to price cash flows
- Arbitrage free, if random variable $\Psi_{0,t} > 0$
- Use exponential affine SDF $\Psi_{0,t}^{(\text{exp.aff})} = \exp(-\sum_{\tau=1}^{t} m_\tau)$
- Assume m_τ is **ergodic stationary** discrete-time random processes
- One-period SDF is linear function $m_\tau = m_\tau(\alpha, \beta) = \alpha + \sum_j \beta_j F_{j,\tau}$ with (not necessarily tradeable) factors F_j
- Moment condition idea to estimate α, β:
  $E[P] = E[\sum_\tau \Psi_\tau \cdot (D_\tau - C_\tau)] = 0$,
  since we expect m_τ to correctly price all PE fund cash flows

Competing models and estimation procedures for m_τ:

Semiparametric Generalized Method of Momemts (GMM):

- Usually applied in SDF framework
- Just requires partially specified model

Parametric Maximum Likelihood:

- Think of m_τ as GLM or GAM, i.e., random variable with parameters of linear form $\mu_{m_\tau} = \alpha + \sum_j \beta_j F_{j,\tau}$
- When m_τ normal distributed, SDF process is discrete-time GBM with time-varying mean (and possibly stdv)

Parametric Bayesian Markov Chain Monte Carlo:

- Log-normal approach of [Ang et al., 2018]
- Does log-normal distribution fit well? Other distribution candidate with additive feature?

3.3 CURRENT ”GMM” APPROACHES

[Driessen et al., 2012] **linear SDF**:

- **cross-sectional ”GMM”** approach with identity weighting matrix
- cross-sectional unit: private vintage portfolio
- estimate SDF on these private vintage portfolios
- asymptotics: let number of funds per portfolio $\to \infty$
- moment conditions suffer from consistency issue, i.e., $\alpha \to \infty$ yields sample moment condition minimum.
- standard errors estimated by cross-sectional bootstrap: randomly select funds of a given vintage to form bootstrap vintage portfolios. requires assumption of cross-sectional independence of funds within a given vintage.
- authors just interested in coefficient estimates, if used to price PE cash flows –> **”Private Market Equivalent”**

[Korteweg and Nagel, 2016] **exponential affine SDF**:

- more ’traditional’ **cross-sectional ”GMM”** approach
- cross-sectional unit: private equity fund
- average over cross-sectional units –> ⚠ just one PE-related ’time-series’
- standard errors are estimated within spatial GMM framework
- estimate SDF on public replication portfolios and use this SDF to price PE fund cash flows –> **(Generalized) Public Market Equivalent**

SDF has to correctly price all horizons $0 \leq h \leq T_i$

GMM-like moment condition for fund (or vintage-portfolio) i and horizon h to specify SDF Ψ:

$$E\left[P_{h,T_i}\right] = 0 \qquad \forall \quad i, h$$

with 'horizon' price (or pricing-error)

$$P_{h,T_i} = \sum_{\tau_i=0}^{T_i} (D_{\tau_i} - C_{\tau_i}) \cdot \frac{\Psi_{0,\tau_i}}{\Psi_{0,h}}$$

Negative relation between α and h, so we need optimal h.

P_{h,T_i} is 'auto-correlated' in both dimensions i, h.

3.5 PRICING ERROR MATRIX (EXAMPLE)

i / h	0	1	2	3	4	5	6
2008	0.60	8.30	1.02	7.04	-7.41	2.74	-1.42
2009	9.22	-6.47	4.50	-0.40	4.77	7.58	-0.71
2010	0.86	-1.91	-4.09	-4.79	4.43	0.09	-5.87
2011	9.14	6.97	0.75	-3.48	-1.76	1.61	-5.09
2012	-8.03	-3.69	-3.06	0.65	6.10	-0.50	
2013	1.14	4.44	-2.63	1.67	-7.20		
2014	0.96	4.07	-3.36	3.38			
2015	-3.27	-4.85	-4.46				
2016	3.21	-4.37					
2017	-4.19						

P_{h,T_i}-matrix for example data as of 2017.

General **extremum estimator** methodology rather than classical GMM framework: $\hat{\alpha}, \hat{\beta} = \text{argmin}_{\alpha,\beta\in\Theta} = \frac{1}{N\cdot(H+1)-\#na}\sum_{i=1}^{N} L_i(\alpha,\beta)$

Empirical loss function for fund/portfolio i

$$L_i(\alpha,\beta) = \sum_{h=0}^{H}\left[P_{h,T_i}\right]^2 \quad (5)$$

$$= \sum_{h=0}^{H}\left[\sum_{\tau_i=0}^{T_i}(D_{\tau_i} - C_{\tau_i})\cdot\frac{\Psi_{0,\tau_i}}{\Psi_{0,h}}\right]^2 \quad (6)$$

$$= \sum_{h=0}^{H}\left[\sum_{\tau_i=0}^{T_i}(D_{\tau_i} - C_{\tau_i})\cdot\exp\left(-\sum_{t=0}^{\tau_i} m_t + \sum_{t=0}^{h} m_t\right)\right]^2 \quad (7)$$

with $m_t = m_t(\alpha,\beta;F_t) = \alpha + \sum_j \beta_j F_{j,t}$

GMM-like estimator with identity weighting matrix:

$\hat{\alpha}, \hat{\beta} = \text{argmin}_{\alpha,\beta \in \Theta} = \frac{1}{H+1} \sum_{h=0}^{H} L_h(\alpha, \beta)$

Empirical loss function for horizon h

$$L_h(\alpha,\beta) = \left[\frac{1}{N}\sum_{i=1}^{N} P_{h,T_i}\right]^2 \tag{8}$$

$$= \left[\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{\tau_i=0}^{T_i}(D_{\tau_i} - C_{\tau_i}) \cdot \frac{\Psi_{0,\tau_i}}{\Psi_{0,h}}\right)\right]^2 \tag{9}$$

$$= \left[\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{\tau_i=0}^{T_i}(D_{\tau_i} - C_{\tau_i}) \cdot \exp\left(-\sum_{t=0}^{\tau_i} m_t + \sum_{t=0}^{h} m_t\right)\right)\right]^2 \tag{10}$$

with $m_t = m_t(\alpha, \beta; F_t) = \alpha + \sum_j \beta_j F_{j,t}$

Penalized empirical loss function

$$L_h(\alpha, \beta) = \left[\frac{1}{N}\sum_{i=1}^{N} P_{h,T_i}\right]^2 + \pi_1 \sum_j |\beta_j| + \pi_2 \sum_j (\beta_j)^2$$

- Lasso: L_1 regularization (variable selection)
- Ridge: L_2 regularization (coefficient shrinkage)
- Elastic-net: combine L_1 and L_2 regularization
- include α in penalty functions?

3.9 CONSISTENCY, ASYMPTOTICS

Asymptotics for vintage year portfolios:

- Fix h, let number of cross-sectional units $i \rightarrow \infty$
- **Consistency**: e.g., convergence in squared mean implies convergence in probability: $E(|\theta - \theta_0|^2) \rightarrow 0$ with $\theta = \alpha, \beta$
- Law of large numbers: average $\rightarrow$ expectation
- Central limit theorem: $(\theta - \theta_0) \cdot C \rightarrow N(0, \Sigma)$ [White, 2000, p. 131]
- Show P_i is stationary process (how for function of θ?)
- Show P_i is strong mixing process (asymptotic independence, which is stronger than ergodicity for stationary processes [White, 2000, p. 48])
- Do we need to specify data-generating process?

4 DATA & MODEL ESTIMATION

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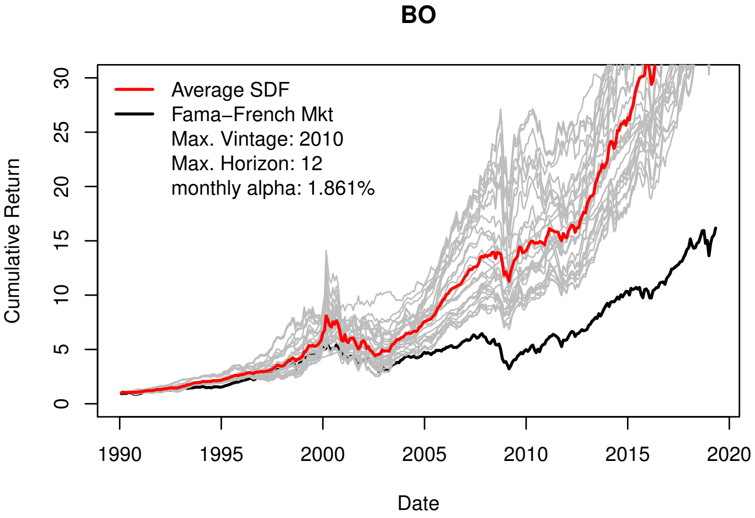
Apply GMM-like methodology to:

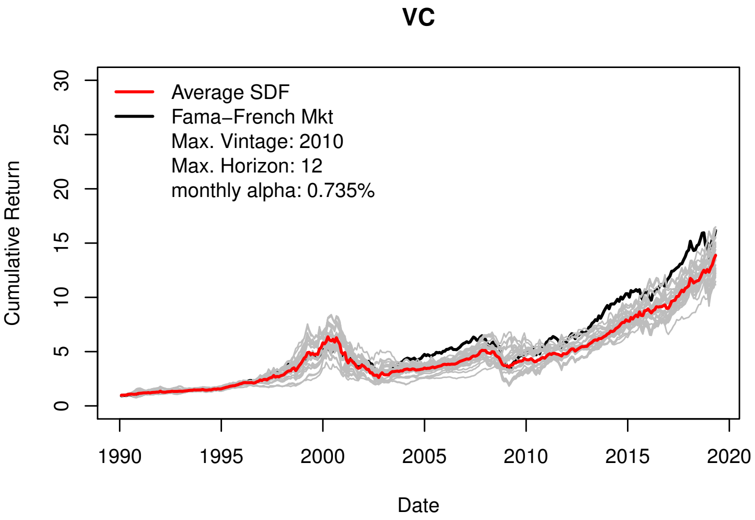
- Preqin cash flow data set as of May 2019
- Use fund data in period 1986-2017Q4
- 'Cross-sectional' unit i: vintage year portfolio, i.e., pool all fund cash flows and valuations of a given vintage into one portfolio (fund size weighted)
- By fund types: buyout (BO), venture capital (VC), private debt (Debt), real assets (Real)
- Maximum horizon 12: $h = 0, 1, 2, ..., 11, 12$ (why not 10 or 15?)
- Maximum vintage 2010: $i = 1986, ..., 2010$ (why not 2005 or 2015?)

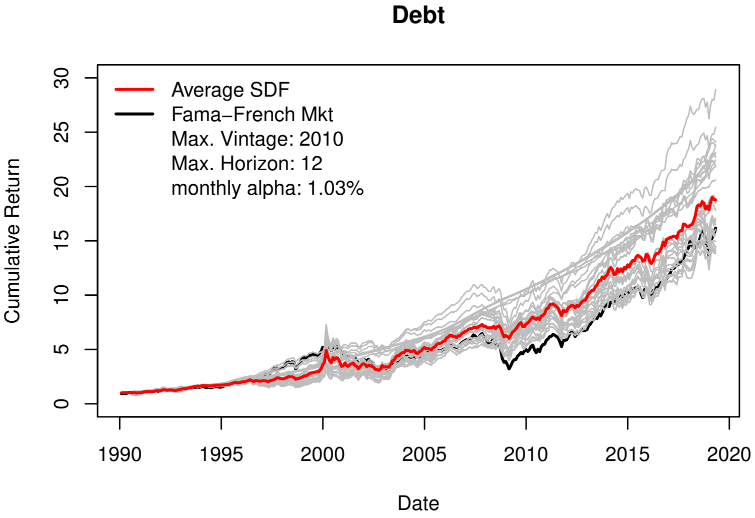
Ensemble of models:

- Use all log-linear Fama-French 5 factor model combinations $j \in \{\text{Mkt-RF, HML, SMB, CMA, RMW}\}$, yields $2^5 - 1 = 31$ models
- $m_t = \alpha + F_{RF,t} + \sum_j \beta_j F_{j,t}$
- Coefficient estimates are very insignificant (also not clear how to correctly calculate coefficient variance matrix)
- Can we linearly combine an ensemble of weak learners to form a stronger one (like in boosting)? Model averaging vs. model selection.

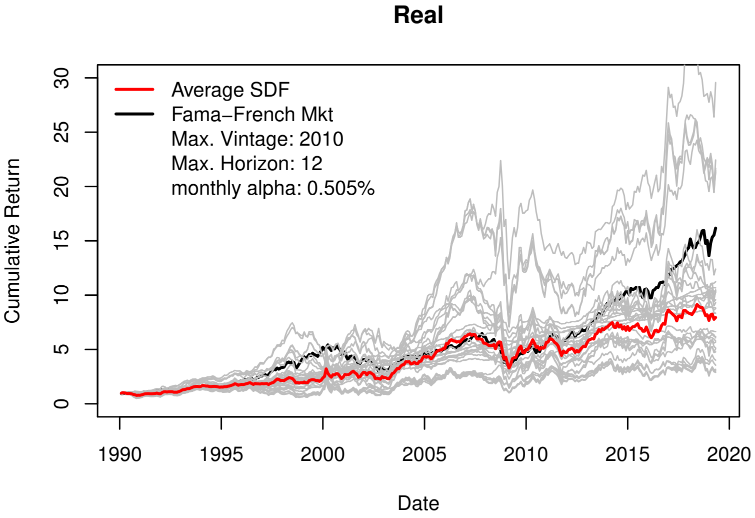

BO
Average SDF
Fama−French Mkt
Max. Vintage: 2010
Max. Horizon: 12
monthly alpha: 1.861%
Cumulative Return
0
5
10
15
20
25
30
1990
1995
2000
2005
2010
2015
2020
Date

4.3 ENSEMBLE OF 31 SDF MODELS: VC

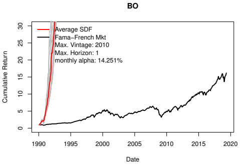



Debt
Average SDF
Fama–French Mkt
Max. Vintage: 2010
Max. Horizon: 12
monthly alpha: 1.03%
Cumulative Return
0
5
10
15
20
25
30
1990
1995
2000
2005
2010
2015
2020
Date

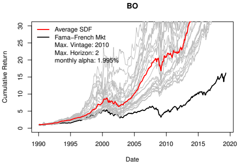
4.3 ENSEMBLE OF 31 SDF MODELS: REAL



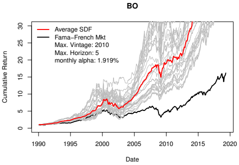
4.3 ENSEMBLE OF 31 SDF MODELS: BO



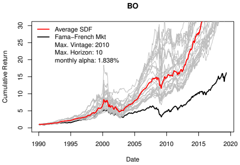
(a) Max. Horizon 1



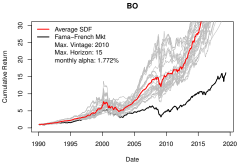
(b) Max. Horizon 2



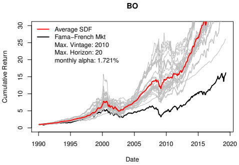
(c) Max. Horizon 5



(d) Max. Horizon 10

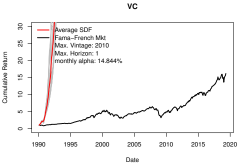


(e) Max. Horizon 15

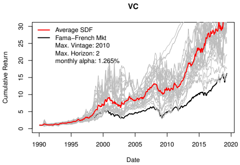


(f) Max. Horizon 20

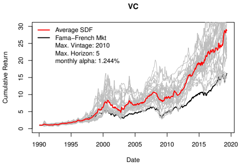
Figure: Buy Out: several max. horizons



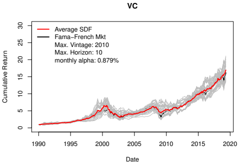
(a) Max. Horizon 1



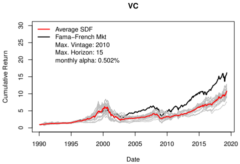
(b) Max. Horizon 2



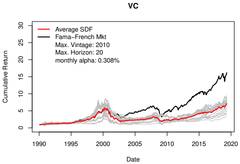
(c) Max. Horizon 5



(d) Max. Horizon 10

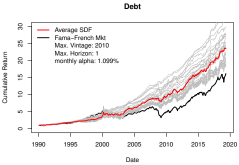


(e) Max. Horizon 15

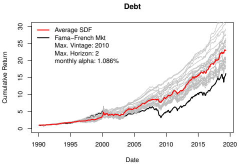


(f) Max. Horizon 20

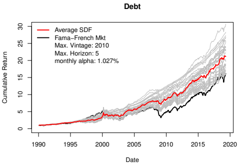
Figure: Venture Capital: several max. horizons



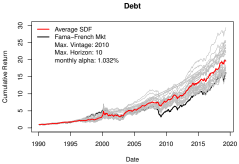
(a) Max. Horizon 1



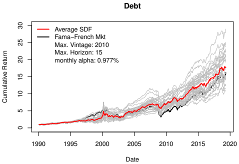
(b) Max. Horizon 2



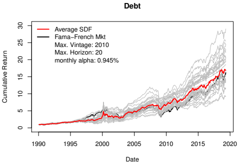
(c) Max. Horizon 5



(d) Max. Horizon 10

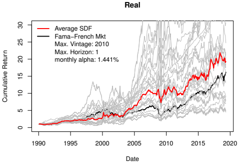


(e) Max. Horizon 15

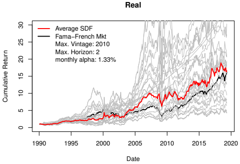


(f) Max. Horizon 20

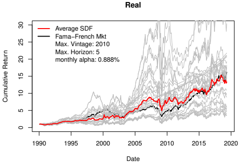
Figure: Private Debt: several max. horizons



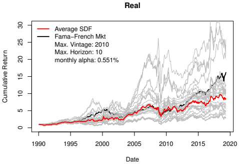
(a) Max. Horizon 1



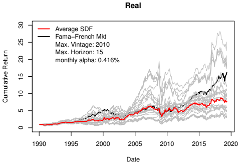
(b) Max. Horizon 2



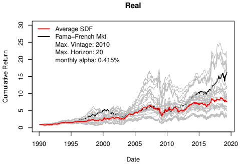
(c) Max. Horizon 5



(d) Max. Horizon 10



(e) Max. Horizon 15



(f) Max. Horizon 20

Figure: Real Assets: several max. horizons

Elastic Net

Compare block vs. leave-one-vintage-out (LOVO) cross validation:

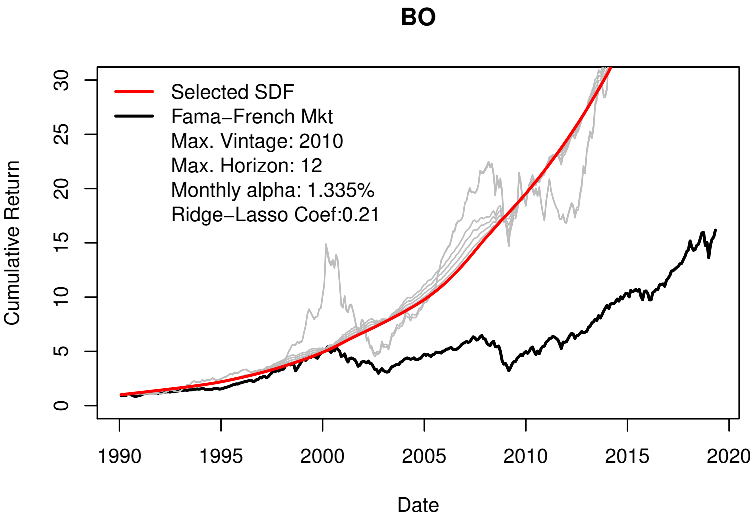
- cross validation for model selection and/or hyper parameter choice in elastic-net π_1 and π_2
- v-block cross validation: consistency requires $n_v/n \to 1$
    - use large validation set compared to training set
    - at best use opposite of leave-one-out cross validation
- h-block cross validation: exclude n_h adjacent observations to alleviate time-dependency
- [Racine, 2000]'s hv-block framework does not include small-sample correction
- Practical compromise: $n_v/n = 0.5$ and no h-blocking

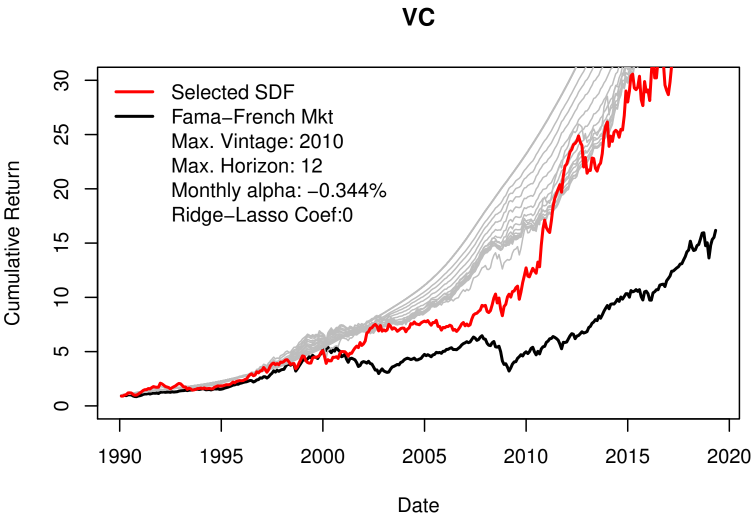
4.5 BLOCK CROSS VALIDATION (EXAMPLE)

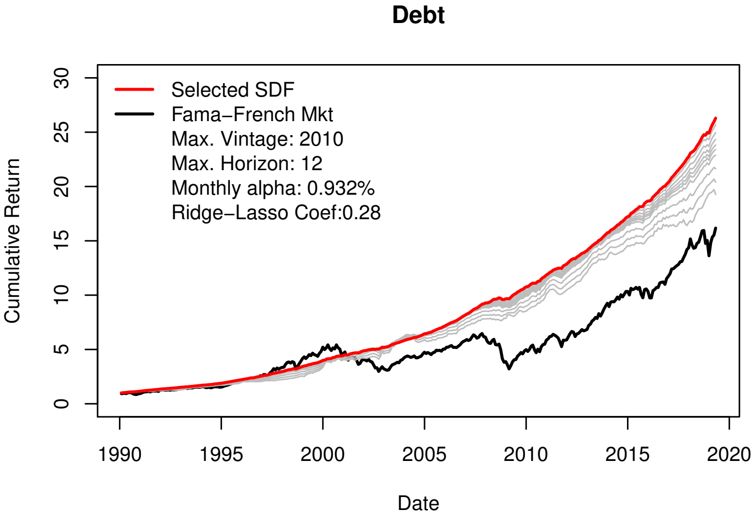
i / h	0	1	2	3	4	5	6
2008	0.60	8.30	1.02	7.04	-7.41	2.74	-1.42
2009	9.22	-6.47	4.50	-0.40	4.77	7.58	-0.71
2010	0.86	-1.91	-4.09	-4.79	4.43	0.09	-5.87
2011	9.14	6.97	0.75	-3.48	-1.76	1.61	-5.09
2012	-8.03	-3.69	-3.06	0.65	6.10	-0.50	
2013	1.14	4.44	-2.63	1.67	-7.20		
2014	0.96	4.07	-3.36	3.38			
2015	-3.27	-4.85	-4.46				
2016	3.21	-4.37					
2017	-4.19						

Data partitions used for training and validation (v-block).
Data adjacent to training set is ignored (h-block).

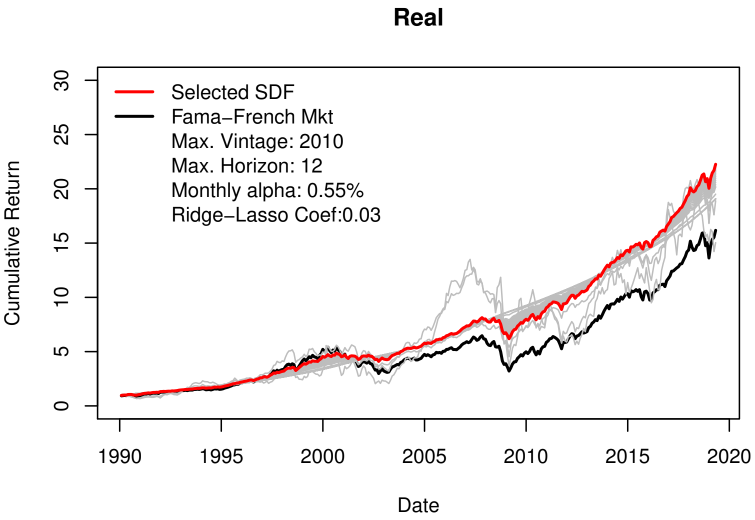
4.6 ELASTIC NET, V-BLOCK CROSS-VALIDATION: BO

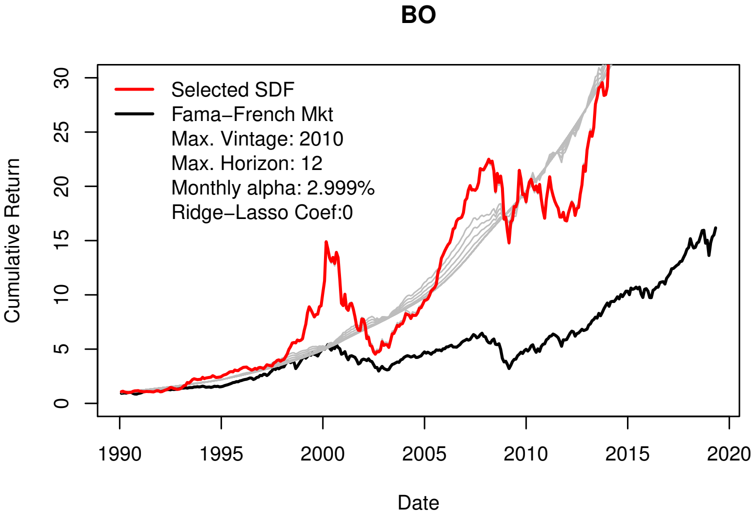



VC
Selected SDF
Fama−French Mkt
Max. Vintage: 2010
Max. Horizon: 12
Monthly alpha: −0.344%
Ridge−Lasso Coef:0
Cumulative Return
0
5
10
15
20
25
30
1990
1995
2000
2005
2010
2015
2020
Date

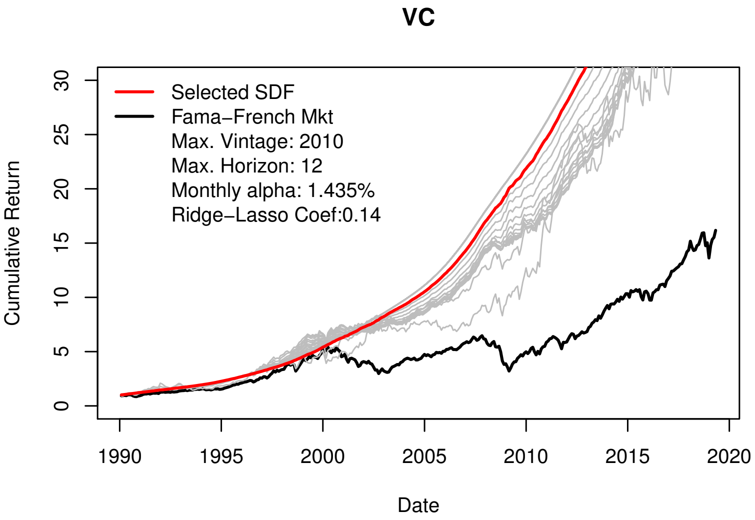

Debt
Selected SDF
Fama–French Mkt
Max. Vintage: 2010
Max. Horizon: 12
Monthly alpha: 0.932%
Ridge–Lasso Coef:0.28
Cumulative Return
0
5
10
15
20
25
30
1990
1995
2000
2005
2010
2015
2020
Date

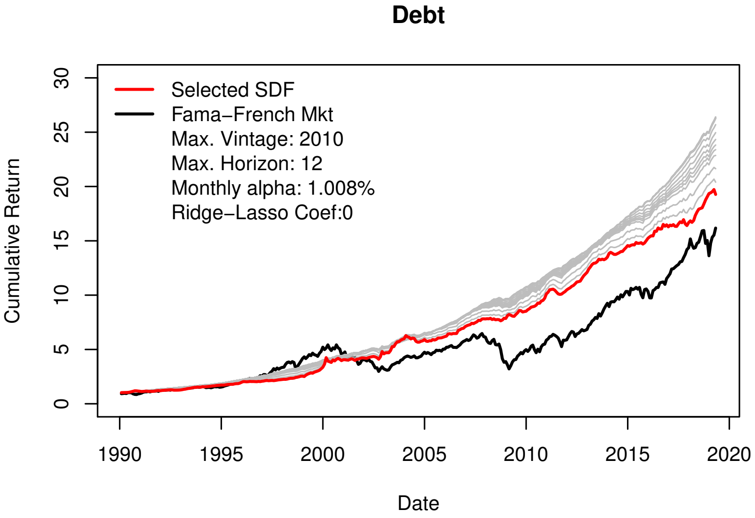
4.6 ELASTIC NET, V-BLOCK CROSS-VALIDATION: REAL

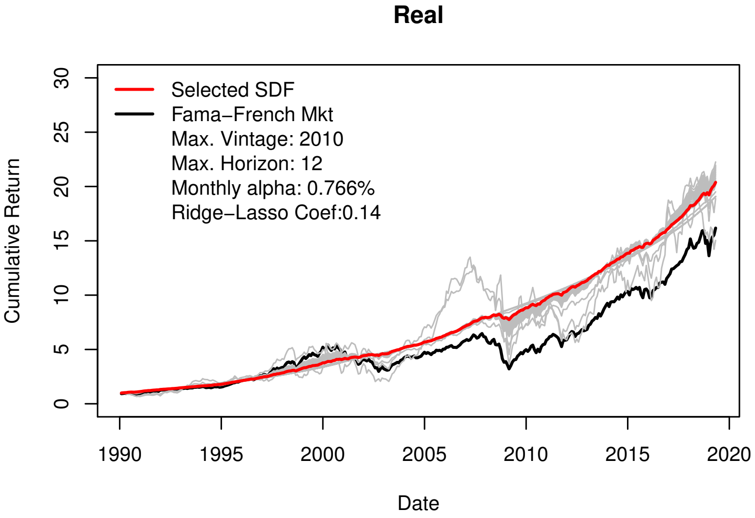



BO
Selected SDF
Fama−French Mkt
Max. Vintage: 2010
Max. Horizon: 12
Monthly alpha: 2.999%
Ridge−Lasso Coef:0
Cumulative Return
0
5
10
15
20
25
30
1990
1995
2000
2005
2010
2015
2020
Date

4.6 ELASTIC NET, LOVO CROSS-VALIDATION: VC




Debt
Selected SDF
Fama–French Mkt
Max. Vintage: 2010
Max. Horizon: 12
Monthly alpha: 1.008%
Ridge–Lasso Coef:0
Cumulative Return
0
5
10
15
20
25
30
1990
1995
2000
2005
2010
2015
2020
Date


Real
Selected SDF
Fama−French Mkt
Max. Vintage: 2010
Max. Horizon: 12
Monthly alpha: 0.766%
Ridge−Lasso Coef:0.14
Cumulative Return
0
5
10
15
20
25
30
1990
1995
2000
2005
2010
2015
2020
Date

5 CONCLUSION

5.1 SUMMARY & OUTLOOK

1. Public Market Equivalent: what do you want?
    - Replication: investable, predictable strategy
    - Pricing: arbitrage-free (positive) SDF model
2. Very hard to establish sound econometric SDF approach for fund-level data and determine significance (time and cross-sectional dependence, non-liquidated funds, most reasonable pricing horizon, small data sample)
3. Horizon moment conditions potentially remedy DLP12's $\alpha \to \infty$ issue and KN16's under-specification issue
4. Extremum estimator vs. GMM-like method
5. Ensemble of models, model selection, cross-validation (elastic net vs. componentwise boosting)

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Working paper and R code
will be available on my blog
Quant-Unit.com
.
Do you have comments?