ON SEMIPARAMETRIC SDF ESTIMATORS

For private equity fund data

Christian Tausch

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Title: On semiparametric Stochastic Discount Factor (SDF) estimators for private equity fund data

Abstract: First, this talk introduces a new spatial SDF estimation framework developed for private equity funds and compares it to similar methodologies. Simulation results suggest that the estimator can improve current approaches, but empirical results remain often insignificant. Second, the talk exhibits how (and why) model combination is used to obtain a strong SDF model from a collection of weak competitors. Consequentially, the empirical model combination results for several private equity fund types appear reasonable.

Agenda

1. Introduction
2. Semiparametric estimation framework (1. paper)
3. Simulation results (1. paper)
4. Model combination (2. paper)
5. Empirical results (2. paper)
6. Conclusion

1 INTRODUCTION

Stochastic Discount Factor Methods for Non-Traded Cash Flows - The Case of Private Equity

Part I Introduction

1. Non-traded cash flows
2. Stochastic discount factors (SDFs)

Part II Numeraire portfolio methods

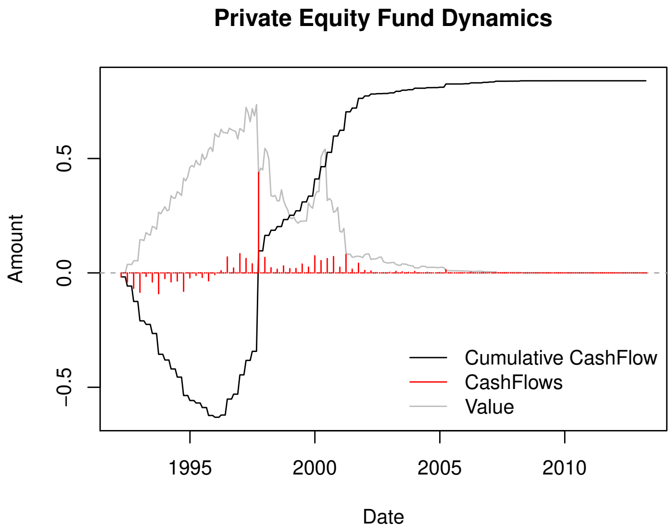
3. Public numeraire equivalent benchmarking
4. Quadratic hedging strategies for private equity fund payment streams

Part III **Semiparametric SDF methods**

5. **A spatial SDF estimator for private equity funds**
6. **The public factor exposure of private equity**

Part IV Parametric SDF methods

7. Risk modeling by parametric SDFs
8. Modeling the exit cash flows of private equity fund investments


Private Equity Fund Dynamics
Amount
0.5
0.0
−0.5
1995
2000
2005
2010
Date
Cumulative CashFlow
CashFlows
Value

Private equity fund $i = 1, 2, \ldots, n$ is characterized by:

Net Asset Value $NAV_{i,t}$ (fund value proxy)

Net Cash Flow $CF_{i,t}$ (fund distributions minus contributions)

Vintage Year V_i (fund inception year)

Public market is given by:

SDF $\Psi_{\tau,t} > 0$ (stochastic discount factor from t to τ)

Risk-free Rate r_t (from period $t - 1$ to t)

Factor Return $F_{j,t} \geq 0$ (zero-net-investment return from $t - 1$ to t)

Time is discrete $t = 1, 2, \ldots, T$.

Stochastic discount factors (SDFs)

- General pricing framework in empirical finance.
- SDFs allow to move cash flows in time.

We can calculate the time-τ price of a time-t cash flow by

$$P_{\tau,t,i} = \mathbb{E}\left[\Psi_{\tau,t} \cdot CF_{t,i}\right] \qquad \forall \quad \tau, t, i \tag{1}$$

where the SDF $\Psi_{\tau,t} = \Psi_{\tau,t}(\theta)$ depends on the parameter vector θ.

If τ and t are both in the past, the realized price is given by

$$P_{\tau,t,i} = \Psi_{\tau,t} \cdot CF_{t,i} \qquad \forall \quad \tau, t, i \tag{2}$$

with $\tau \leq t$ or $\tau \geq t$.

2 Semiparametric Estimation Framework

2.1 SEMIPARAMETRIC ESTIMATORS

- Empirical asset pricing usually uses **semiparametric** approaches to determine the 'optimal' parameter vector θ of the SDF $\Psi_{\tau,t}(\theta)$.
- Semiparametric means we impose **no distributional assumptions** on the random variable Ψ.
- The parameter vector θ contains no distributional parameters (like μ, σ for a normal distribution).
- We want to **parsimoniously** explain asset returns (cash flows).
- When testing SDFs, we want to test if a given SDF satisfactorily prices the assets and not if the asset returns are (e.g.) normally distributed.
- Parametric estimation (like maximum likelihood) is usually more **efficient** (unbiased with smaller variance) when we know the underlying distribution.

The fund i pricing error at time τ is defined as

$$\epsilon_{\tau,i} = \sum_{t=1}^{T} P_{\tau,t,i} = \sum_{t=1}^{T} \Psi_{\tau,t} \cdot CF_{t,i} \qquad \forall \quad \tau, i \tag{3}$$

The w_i-weighted and $\mathcal{T}_i$-averaged fund pricing error is defined as

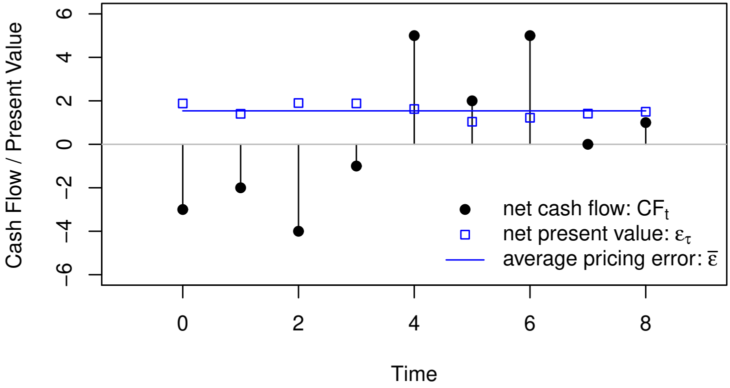
$$\bar{\epsilon}_i = w_i \cdot \frac{1}{\text{card}(\mathcal{T}_i)} \sum_{\tau \in \mathcal{T}_i} \epsilon_{\tau,i} \qquad \forall \quad i \tag{4}$$

where $\mathcal{T}_i$ is the set of relevant net present value dates for fund i.

To find θ, our LMD estimator minimizes the average square of $\bar{\epsilon}$

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \; S_n(\theta) \quad \text{with} \quad S_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} (\bar{\epsilon}_i)^2 \tag{5}$$

- The time index t is relevant for the net cash flows (black dots).
- The time index τ is used for the net present values (NPVs) of this net cash flow stream (blue boxes).
- The weighted average of these net present values gives the average pricing error $\bar{\epsilon}$ as defined in equation 4 (solid blue line).



[Driessen et al., 2012] just include fund inception date in $\mathcal{T}_i$.

[Korteweg and Nagel, 2016] replace $S_n(\theta)$ in equation 5 by

$$S_n(\theta) = \left(\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i\right)^2.$$

	[Driessen et al., 2012]	[Korteweg and Nagel, 2016]	Our approach
M-estimator	Least-Mean-Distance	Generalized Method of Moments	Least-Mean-Distance
Pricing error averaging	No	No	Yes
Cash flows priced	PE cash flows	public cash flows	PE cash flows
Asymptotics	cross-sectional #funds $\to \infty$	time-series #vintages $\to \infty$	spatial # of both $\to \infty$
Inference	bootstrap	spatial HAC	cross-validation & spatial HAC
Cross-sectional unit	vintage year portfolio	single fund	testing both
SDF	simple linear	exponentially affine	testing both

Table: Comparison to similar estimation frameworks.

2.5 UNDERLYING ASSUMPTIONS

Formal SDF estimation framework based on 5 assumptions:

1. Vintage year portfolios (VYPs): group all n funds into VYPs.
2. Vintage year asymptotics: $n \to \infty$ as $V \to \infty$.
3. Law of large numbers: $n^{-1} \sum_{i=1}^{n} \bar{\epsilon}_i \overset{a.s.}{\to} E[\bar{\epsilon}] \quad \text{as} \quad V, n \to \infty$.
4. Consistency: $\hat{\theta} \overset{p}{\to} \theta_0$ as $V, n \to \infty$. $E[\bar{\epsilon}] = 0$ if and only if $\theta = \theta_0$.
5. Central limit theorem: $\sqrt{n}(\hat{\theta} - \theta_0) \overset{d}{\to} \mathcal{N}(0, \Sigma)$ as $V, n \to \infty$ with covariance matrix Σ characterized according to [Pötscher and Prucha, 1997, Theorem 11.2.b, Theorem H.1].

On this basis, we derive (spatial) asymptotic inference framework that was missing in [Driessen et al., 2012]. Spatial notion was introduced by [Korteweg and Nagel, 2016].

3 SIMULATION RESULTS

We test a **simple linear** SDF model as in [Driessen et al., 2012]

$$\Psi^{\mathrm{SL}}_{\tau,t}(\theta) = \prod_{h=1}^{t} \left(1 + \alpha + r_h + \sum_j \beta_j F_{j,h}\right)^{-1} \prod_{h=1}^{\tau} \left(1 + \alpha + r_h + \sum_j \beta_j F_{j,h}\right) \quad (6)$$

and an **exponential affine** SDF model adapted from [Korteweg and Nagel, 2016]

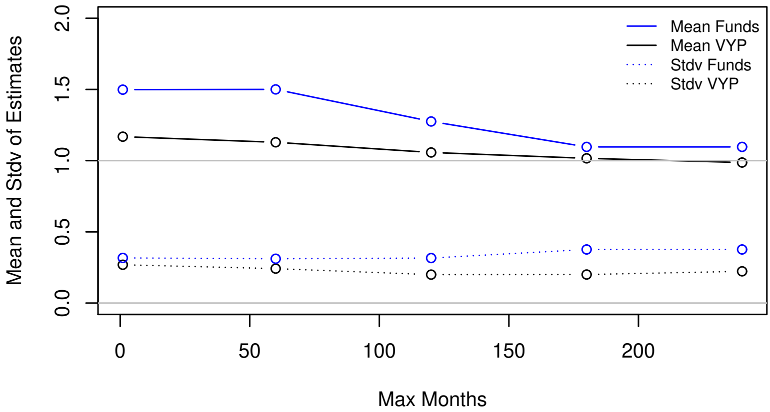
$$\Psi^{\mathrm{EA}}_{\tau,t}(\theta) = \exp\left[- \sum_{h=\tau}^{t} \left(\alpha + \log(1 + r_h) + \sum_j \beta_j \cdot \log(1 + F_{j,h}) \right) \right] \quad (7)$$

with (arithmetic) risk-free return r, (arithmetic) zero-net-investment portfolio returns F_j, and parameter vector $\theta = (\alpha, \beta)$.

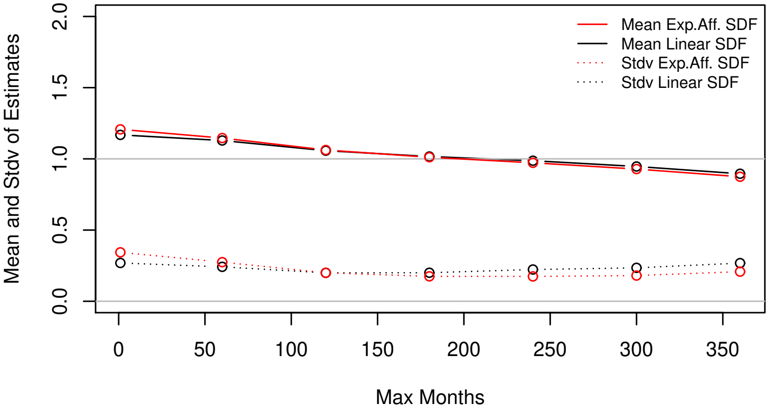
For 20 vintage years, simulate typical private equity funds that each enters and exits 15 deals within their lifetime of 15 years. Exit cash flows are driven by simple α and β_{MKT} factor models using realized market returns. To analyze the following questions:

1. Is it beneficial to use vintage year portfolios (VYPs) instead of individual funds?
2. Which SDF model performs better when we also use the corresponding data generating process (i.e., assume correct model specification)?
3. How is estimator precision affected by varying numbers of vintage years and cross-sectional units?
4. Which is the optimal set of present value times $\mathcal{T}$?

1. Is it beneficial to use vintage year portfolios (VYPs) instead of individual funds?



2. Which SDF model performs better?
4. Which is the optimal set of present value times $\mathcal{T}$?



3. How is estimator precision affected by varying numbers of vintage years and cross-sectional units?

	Base	Big n/V	Big V	Big V	Small V	Small V
Start vintage	1986	1986	1967	1967	1986	1996
End vintage	2005	2005	2005	2005	1995	2005
#Funds per vintage	20	40	10	20	20	20
Mean β_{MKT}	1.011	1.020	0.993	1.015	1.027	0.934
Stdv β_{MKT}	0.187	0.133	0.263	0.227	0.232	0.418

Table: Simulation study for varying number of vintages and number of funds per vintage. We use vintage year portfolios, the simple linear SDF with true $\beta_{\mathrm{MKT}} = 1$, maximum month 180, and 500 simulation iterations.

3.3 SIMULATION RESULTS (SUMMARY)

1. Is it beneficial to use vintage year portfolios (VYPs) instead of individual funds?

   Yes, $\hat{\beta}_{\mathrm{MKT}}$ is 1.016 (0.2) for the vintage year portfolio and 1.096 (0.376) for individual funds; with true $\beta_{\mathrm{MKT}} = 1$.

2. Which SDF model performs better when we also use the corresponding data generating process (i.e., assume correct model specification)?

   Both exhibit similar small-sample bias and variance.

3. How is estimator precision affected by varying numbers of vintage years and cross-sectional units?

   Increasing the number of funds per vintage year portfolio appears to decrease the estimator's variance more effectively than adding more vintage years. Bias always very similar.

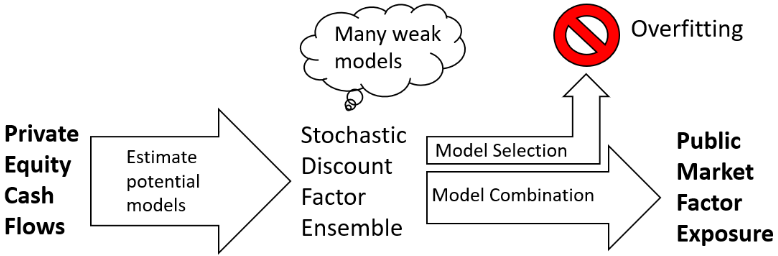
4. Which is the optimal set of present value times $\mathcal{T}$?

   Use all dates/quarters within the fund lifetime.

3.4 REMAINING CHALLENGES

- Simulation results indicate high small-sample variance even for simple data generating processes.
- Empirical estimation for simple two-factor models reveals (using public q^5-investment factors of [Hou et al., 2020]):
  - ⚠ very high asymptotic standard error estimates (for vintage year portfolios),
  - hv-block cross-validation standard errors are smaller,
  - model selection remains challenging when confronted with a large set of competing models.
- 💡 Model combination instead of model selection?

4 MODEL COMBINATION



Transform weak model ensemble to strong multi-factor model.

4.2 WHY SO MANY/WEAK/DIFFICULT?

- Why so many models?
  - Many public market factor candidates
  - Many potential estimators, loss functions, hyperparameters
  - Many different proprietary private data sets
- Why so weak models?
  - Sparse private equity fund data ($\leq$ 40 vintages)
  - Near-epoch dependency by overlapping fund cash flows
  - Multi-factor models almost surely overfit
- Why model selection is difficult?
  - Model uncertainty especially high for weak models
  - Limited data may encourage data snooping
  - Correct post model selection inference is generally hard

The weighted pricing error obtained by SDF model averaging is defined as

$$\epsilon_{\tau,i}^{(M^*)} = \sum_{m=1}^{M^*} w_m \sum_{t=1}^{T} \Psi_{\tau,t}^{(m)} CF_{t,i} \tag{8}$$

with model weight $w_m \geq 0$ and all weights sum to one $\sum_m^{M^*} w_m = 1$. The ensemble size is M^*.

- Forecast combination puzzle: Often $w_m = \frac{1}{M*}$ outperforms more 'advanced' weighting schemes.
- Model combination can be perceived as diversification strategy to minimize the risk of selecting an invalid model (i.e., investing everything in the wrong replication strategy).

5 Empirical Results

Estimate four two-factor models (using linear SDF from equation 6):
MKT-RF $\times$ {SMB or HML or HDY-MKT or QLT-MKT} with

- **MKT-RF:** MSCI Market Return Minus Risk-free Rate
- **SMB:** MSCI Small Cap Minus MSCI Large Cap Return
- **HML:** MSCI Value Minus MSCI Growth Return
- **HDY-MKT:** MSCI High Dividend Yield Minus MSCI Market Return
- **QLT-MKT:** MSCI Quality Minus MSCI Market Return

For each of these four models, we generate $2 \times 2 \times 5$ estimates by varying (i) a quadratic and last absolute deviance loss function, (ii) equal- and fund-size-weighted cash flows, and (iii) maximum months in {120, 150, 180, 210, 240} for $\mathcal{T}$.

Finally, simply **average** the $4 \times 2 \times 2 \times 5$ model **coefficients**.

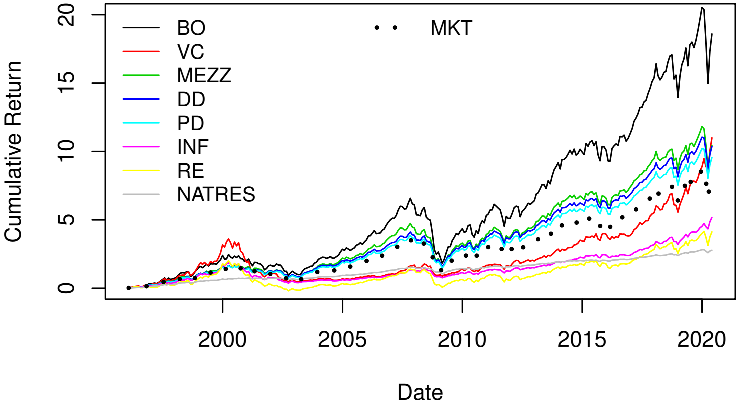
The public factor exposure of private equity

Type	MKT-RF	HML	SMB	HDY-MKT	QLT-MKT
BO	1.33 (0.15)	-0.15 (0.12)	0.2 (0.03)	0.3 (0.1)	0.21 (0.05)
DD	0.96 (0.09)	-0.11 (0.04)	0.21 (0.01)	0.14 (0.1)	0.16 (0.05)
INF	0.71 (0.22)	-0.37 (0.06)	-0.33 (0.13)	-0.47 (0.35)	0.36 (0.11)
MEZZ	1.08 (0.13)	0.06 (0.1)	0.14 (0.04)	0.16 (0.1)	0.06 (0.11)
NATRES	0.36 (0.27)	-0.04 (0.22)	-0.02 (0.22)	0.16 (0.36)	0.11 (0.17)
PD	0.96 (0.08)	-0.07 (0.04)	0.16 (0.03)	0.06 (0.09)	0.15 (0.04)
RE	1.14 (0.44)	-0.3 (0.16)	-0.42 (0.13)	-0.91 (0.15)	-0.4 (0.1)
VC	1.02 (0.67)	-0.61 (0.11)	-0.42 (0.03)	-0.75 (0.14)	0.84 (0.61)
MKT	1	0	0	0	0

Table: Multivariate five-factor models obtained by simple coefficient averaging (with standard deviations in parenthesis).

Private equity: Preqin cash flow data. Public: MSCI style indices.

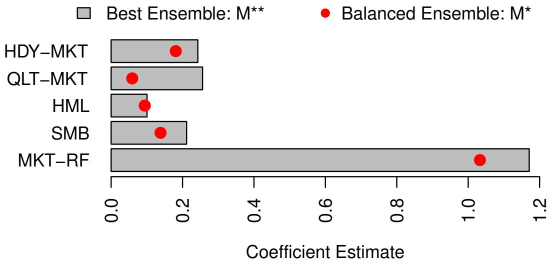
Did private equity outperform the public market portfolio?



Type	Annualized Return				Sharpe Ratio
	mean.R	stdv.R	mean.R-RF	stdv.R-RF	mean/stdv.R-RF
BO	**0.152**	0.195	**0.125**	0.196	0.641
DD	0.116	0.144	0.091	0.144	0.630
INF	0.085	0.119	0.060	0.119	0.506
MEZZ	0.120	0.162	0.094	0.162	0.581
NATRES	0.057	**0.049**	0.033	**0.049**	**0.671**
PD	0.113	0.143	0.087	0.143	0.610
RE	0.092	0.203	0.067	0.203	0.329
VC	0.124	0.176	0.099	0.176	0.561
MKT	0.107	0.152	0.082	0.152	0.536

Table: Annualized average returns, standard deviations (annualized by the square root of time formula), and Sharpe ratios (i.e., the ratio of mean.R-RF to stdv.R-RF) implied by the five-factor models (1996-01-31 to 2020-05-31).

Bottom-up (fund-by-fund) aggregation of averaged coefficients
for a sample portfolio of 100 private capital funds.



- Balanced Ensemble: all valid SDF models for a given fund.
- Best Ensemble: subset of all valid SDF models with smallest pricing error for a given fund.

6 CONCLUSION

- Significant semiparametric SDF estimates for private equity funds are hard to obtain. Asymptotic inference not very useful when forming vintage year portfolios.
- Model combination is a straightforward means to form a strong(er) SDF model from a collection of weak competitors.
- Conjecture: Averaging pricing errors over cash flow duration (fund lifetime) may be general feature of an 'optimal' SDF estimator for non-traded cash flows.
- Future research: Effect of taking historical (fixed) public market returns vs simulated scenarios in simulation study: What are the issues? What is optimal?
- Future research: Analyze improved version of the [Korteweg and Nagel, 2016] estimator (simulation-based portfolios avoid under-identification, but compatible with averaging pricing errors?).

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Working paper and R code
will be available on my blog
Quant-Unit.com

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Do you have comments?