Modeling the exit cash flows of private equity fund investments

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March 21, 2022

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Abstract

Risk perception in private equity is notoriously difficult, as the cash flow patterns associated with private capital funds are not well understood on the underlying deal level. This paper analyzes the realized exit cash flows of individual portfolio companies in a *joint* modeling framework that describes *both* the exit timing *and* exit performance. Particularly we choose an exit timing model suited for the interval-censored nature of private equity deal data and an exit (performance) multiple approach appropriate for the high numbers of company defaults observed in private equity. The corresponding parametric joint model is estimated using maximum likelihood for a Buyout and Venture Capital dataset and applied within a Monte Carlo simulation example to demonstrate the suitability of our approach in the risk management context. Here especially undiversified private equity fund investors benefit from the improved insights offered by risk analysis tools that can incorporate detailed company-level information.

Keywords

Private equity, Exit behavior, Cash flow risk simulation, Buyout, Venture Capital.

Key messages

- This paper develops a deal-level model for the exit cash flows of private equity funds.
- The model jointly describes exit timing and performance of portfolio companies.
- Deal-level data for Buyout and Venture Capital funds are used for empirical estimation.
- A Monte Carlo simulation example analyses risk management applications.

Title

Modeling the exit cash flows of private equity fund investments

1 Introduction

Private capital funds control financial assets with a cumulative valuation of \$4.99 trillion (as of September 2019; Source: Preqin). Fund investors naturally ask: (i) what will happen with these assets in the future, or, on a small scale, (ii) what is the economic hazard associated with a particular private equity fund stake? Fundamentally, (iii) how much cash will be realized and (iv) when? These questions arise because a Private Equity Fund (PEF) is generically constructed as a limited partnership with a bounded lifetime that is not tradable on public markets[1]. The fund manager receives an unfunded upfront commitment from fund investors and then controls the timing of all discretionary investment and divestment cash flows. Fund investors can consider these cash flows as the outcome of exogenous random variables.

This article follows the perception that a *universal* risk measurement methodology tailored for the illiquid nature of PEF investments shall operate on the underlying *deal level*. Particularly, undiversified fund investors benefit from company-level models that achieve to include very granular (deal-level) information in addition to common fund-level characteristics. However, little is known about the dynamic behavior of single private equity fund investments. Especially, the interaction between holding period and total return that eventually determines the cash flows to investors is not well grasped yet. The empirical and theoretical private equity literature lacks comprehensive deal-level concepts that jointly describe both (i) the fund manager's endogenous timing of cash flows and (ii) the risk and return of the underlying fund holdings. These highly related aspects are, unfortunately, often studied in quite fragmented approaches.

To better understand the exit behavior of private equity fund investments on portfolio-company-level, we analyze the connection between (i) exit timing and (ii) return of underlying fund assets by two parametric models (for the marginal distributions) linked by a copula. Additionally, the exit performance model is conditioned on exit timing. The exit timing regression is based on a parametric multiplicative hazard rate formulation adapted for time-variant covariates. The return multiple regression employs a so-called two-part or hurdle model to account for the zero-heavy nature of historically observed PEF deal returns (Min & Agresti, 2002). In summary, the dependency between exit timing and exit multiple is generated by three means: (i) using the same set of public market covariates in both marginal models, (ii) using exit timing as an independent variable in the exit multiple regression,

[1] Kaplan & Strömberg (2009) describe the nature and economics of PEFs in more detail.

and (iii) using a copula to model possible remaining dependencies.

The access to a proprietary deal-level dataset of Buyout (BO) and Venture Capital (VC) fund investments allows our modeling idea's empirical application. First, the aforementioned exit timing and return multiple regression models are estimated by maximum likelihood for both datasets. Here the asset-level granularity permits the inclusion of many covariates that are not available in fund-level regressions. Second, the estimated models are applied in a Monte Carlo simulation example that, with its undiversified PEF portfolio setting, indicates the advantages of deal-level over fund-level cash flow simulation approaches.

The article is organized as follows: Section 2 reviews related literature. Section 3 presents a joint model for exit timing and exit return multiple that can be estimated by maximum likelihood. Section 4 reports the empirical results of this regression approach for a BO and VC dataset. Section 5 discusses a risk management application of the model estimates in a Monte Carlo simulation example. Section 6 finally concludes.

2 Related literature

2.1 Empirical private equity analyses

Published empirical analyses on deal level focus *either* on the exit route *or* on the asset performance of private equity investments. Unfortunately, joint empirical analyses of both aspects only exist as unpublished working papers (Das et al., 2002; Ljungqvist & Richardson, 2003).

The realized exit routes of VC fund investments (e.g., initial public offering, trade sale, liquidation) are analyzed by Giot & Schwienbacher (2007) and Félix et al. (2014) utilizing competing risk models. Jenkinson & Sousa (2015) employ a multiplicative hazard model and a trinomial logistic regression for a BO dataset. Cumming (2008) and Schmidt et al. (2010) use multinomial logit models in similar VC and BO studies. Cumming et al. (2014) survey the firm-level exit performance of governmental and independent VC investments in Europe. All exit route regressions solely incorporate static, time-invariant covariates.

The market timing abilities of PEF managers are analyzed by Jenkinson et al. (2018) in the context of EBITDA market multiple expansion for both the entry and exit dates of fund investments. Gredil (2018) examines fund managers' abilities to predict unfavorable public market returns following deal exits[2].

The return and risk of VC companies are studied by Cochrane (2005) and Korteweg & Sorensen (2010). They develop sample selection correction methodologies[3] that allow the calculation of the amended return of VC investments from observed financing round valuation data. Both approaches rely on log-normally distributed returns. The value creation of BO firms is examined by Guo et al. (2011) and Valkama et al. (2013). Here the return drivers of BO investments are identified in detailed deal-level regressions.

2.2 Stochastic private equity models

The first private equity *fund-level* model that primarily relies on Gaussian stochastic processes is introduced by de Malherbe (2004). Buchner (2017) utilizes a similar framework to distinguish between fund-level (i) market, (ii) liquidity, and (iii) cash flow risk. Buchner et al. (2010) propose a stochastic model on the typical cash flow dynamics of private equity funds that solely relies on observable cash flow data. Conclusive from a diversified portfolio perspective, these approaches inherently tend to neglect asset-level characteristics.

Further, there exist some structural *deal-level* models that are designed to address particular private-equity-related questions. Bongaerts & Charlier (2009) estimate the capital requirements for private equity investments under Basel II. Braun et al. (2011) quantify the risk appetite of BO fund managers. Escobar et al. (2011) examine the portfolio optimization problem for private equity investors. Dong et al. (2012) assess the credit risk associated with a portfolio of private infrastructure projects. Lahmann et al. (2016) focus on the stepwise debt reduction associated with BO investments. Each paper deals with private equity specific issues; however, the general structural (default) framework applied therein is explicitly developed rather for public debt than for private equity. The biggest structural drawback associated with these approaches is their assumption of a deterministic exit timing (at bond maturity when no default happens before).

2.3 Structural vs reduced form approach

Modeling private equity deal exits is similar to modeling loan defaults. In both instances, we are interested in an event time (of exit and default, re-

[2] We thank an anonymous referee for mentioning the connection to the market timing literature.

[3] Their sample selection correction methodologies, that model the probability of observing further deal valuations, could be theoretically used to model exit timing.

spectively) and the cash flow that results at that time. The credit risk literature distinguishes between structural and reduced form modeling approaches. Jarrow & Protter (2004) outline their differences from an information based perspective: ‘Structural models assume complete knowledge of a very detailed information set, akin to that held by the firm’s managers. [...] In contrast, reduced form models assume knowledge of a less detailed information set, akin to that observed by the market’. As a consequence, reduced form model formulations naturally arise in some incomplete (partial/imperfect/latent/noisy) information settings (Giesecke, 2006). Jarrow (2009) further points out that ’Reduced form models take as exogenous both the firms default time and its recovery rate process [...] this results in simplified computational procedures’.

We suggest that reduced form models constitute a fruitful alternative to existing structural PEF models since incomplete information settings can be assumed characteristic for private equity investments. Additionally, structural modeling of PEF deals may be exacerbated by complex and dynamic capital structures and endogenous default thresholds; see Leland & Toft (1996) for a seminal public market case, and Forte & Lovreta (2012) for a more recent approach. These traditional endogeneity concerns may be further aggravated by fund manager incentives that arise in typical PE settings, i.e., existing fee agreements or fundraising for new funds could influence the exit behavior (Robinson & Sensoy, 2013; Barber & Yasuda, 2017; Hüther et al., 2020). Therefore, structural models need to incorporate rules that formalize the endogenous bankruptcy, trade sale, and IPO decision making. Generally, reduced form models strive to describe the data but not necessarily the underlying cause and effect phenomenon like structural approaches. Thus, reduced form models for PE deals are computationally cheaper than structural models since they avoid unnecessary overhead like simulating intermediate deal value paths between entry and exit.

In summary, considering deals as ’black boxes’ should be usually easier, faster, and more realistic for a typical fund investor than relying on structural modeling. We can regard the model proposed in the following section 3 as reduced from approach on deal level.

3 Parametric exit dynamics model

3.1 Simple probability space

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a filtered probability space in continuous-time (satisfying the usual hypotheses) with sample space Ω, $\mathscr{F}$ a σ-algebra of subsets of Ω,

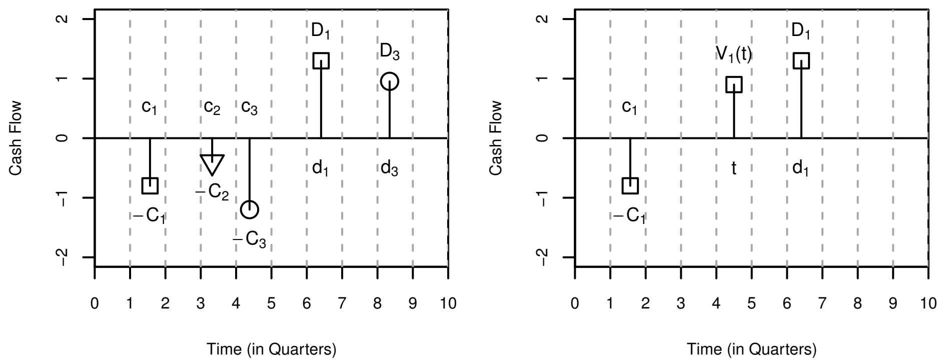

Figure 1: Visualization of fund cash flows and valuations. In the left chart, we can observe three entry events, but the exit event (d_3, D_3) is right-censored. The right chart illustrates an intermediate valuation $V_1(t)$ for the sale cash flow (d_1, D_1).

and the real world probability measure $\mathbb{P}$. The corresponding filtration is given by $(\mathscr{F}_t)_{t\in[0,T^*]}$ and $\mathscr{F} = \mathscr{F}_{T^*}$ for simplicity. All subsequently defined random variables are assumed to be $\mathscr{F}$-measurable.

We consider a private equity fund that holds $n \in \mathbb{N}$ company investments at a given point in its lifetime $t \in [0, T^*]$. Each company investment (deal) is characterized by its current net asset value $V_{i,t} \geq 0$ and the initial entry event (c_i, C_i) where c_i denotes the entry date of the ith investment and $C_i > 0$ denotes the purchase price of the ith asset with $i = 1, 2, ..., n$. We are now interested in a joint statistical model for the exit event (d_i, D_i) where d_i denotes the exit date and $D_i \geq 0$ denotes the sale price of the ith company. To incorporate additional information into our regression analysis we introduce the covariate random vector $X_{i,t} \in \mathbb{R}^m$ that can contain e.g., public market information or company details. Since we know that the fund holds n investments at time t it follows $c_i < t < d_i$ for all $i = 1, 2, ..., n$. To assure $c_i < d_i$ we construct the exit timing by $d_i = c_i + T_i$ where T_i is a positive random variable. Figure 1 visualize the probability space components (c, C), (d, D), and V.

3.2 Marginal distributions of Multiple and Timing

The dependent variables of our joint regression approach are defined as exit Multiple variable $Y_i = D_i/V_{i,t}$ and the exit Timing variable $y_i = d_i - t$.

Due to the semicontinuous (zero-heavy) nature of the MULTIPLE $Y \in \mathbb{R}_{\geq 0}$, our two-part models splits the univariate cumulative distribution function (CDF) and probability density function (PDF) into a point mass at zero and an absolutely continuous part (Min & Agresti, 2002)

$$\begin{array}{rcccl} F_Y(\bar{Y}) & = & \mathbb{P}\left[\frac{D}{V(t)} \leq \bar{Y}|\mathscr{F}_t\right] & = & \pi_0(\mathbf{X})\mathbf{1}_{\{\bar{Y}\geq 0\}} + [1-\pi_0(\mathbf{X})]G_Y\left(\bar{Y}|\mathbf{X}, \xi_Y\right) \\ f_Y(\bar{Y}) & = & \frac{\delta}{\delta \bar{Y}} F_Y(\bar{Y}) & = & \pi_0(\mathbf{X})\mathbf{1}_{\{\bar{Y}=0\}} + [1-\pi_0(\mathbf{X})]g_Y\left(\bar{Y}|\mathbf{X}, \xi_Y\right) \end{array}$$

where the default probability $\pi_0(\mathbf{X})$ is conditioned on some covariates $\mathbf{X}$, and $\mathbf{1}_{\{\}}$ denotes the indicator function. G_Y and g_Y represent a continuous CDF and PDF, respectively, that are conditioned on vectors of covariates $\mathbf{X}$ and parameters ξ_Y. Further the TIMING $y \in \mathbb{R}_{>0}$ is specified by the conditional survival model

$$S_y\left(\bar{y}|t\right) = \mathbb{P}\left[d > t + \bar{y}|\mathscr{F}_t\right] = \frac{S_y\left(t+\bar{y}|\mathbf{X}, \xi_y\right)}{S_y\left(t|\mathbf{X}, \xi_{\mathbf{y}}\right)}$$

where S_y denotes an absolutely continuous survival function with covariate vector $\mathbf{X}$ and parameter vector ξ_y. Due to interval censoring, introduced by the quarterly reporting practice common in the private equity industry, the TIMING density function is linearly approximated by rectangle probabilities

$$f_y(\bar{y}|t) = S_y\left(\bar{y}|t\right) h_y\left(t+\bar{y}\right) \approx \frac{S_y\left(\bar{y}-\Delta|t\right) - S_y\left(\bar{y}|t\right)}{\Delta}$$

where h_y is the corresponding hazard function and $\Delta = 0.25$ for quarterly observations.

3.3 Bivariate copula model

We define the bivariate distribution in terms of the distribution function for Y and survival function for y

$$F_{Yy}(\bar{Y}, \bar{y}|t) = \mathbb{P}\left[\frac{D}{V(t)} \leq \bar{Y}, d > t + \bar{y}\,\middle|\, t, \mathscr{F}_t\right]$$

Although Shi et al. (2015) favor a parametric copula model over a straightforward conditional probability decomposition in a similar insurance setting, we opt to combine both approaches: First we include TIMING in the predictor set $d \in \mathbf{X}$ for both complementary two-part models, π_0 and G_Y, and use $\mathscr{F}_d$-information for all covariates $\mathbf{X}$. Next, we assume conditional independence between the MULTIPLE and TIMING model while allowing for dependency in their residuals via the parametric copula function $\mathrm{Cop}(u,v)$. In this case, the CDF is constructed as

$$\begin{array}{lcllcl}
F_0 & := & F_{Yy}(\bar{Y}, \bar{y}|t, \bar{Y}=0) & = & 1 - S_y\left(\bar{y}|t\right) \\
F_1 & := & F_{Yy}(\bar{Y}, \bar{y}|t, \bar{Y}>0) & = & \mathrm{Cop}[G_Y(\bar{Y}), S_y\left(\bar{y}|t\right)] \\
F_{Yy} & := & F_{Yy}(\bar{Y}, \bar{y}|t, \bar{Y}\geq 0) & = & \mathbf{1}_{\left\{\bar{Y}=0\right\}} F_0 \pi_0 + \mathbf{1}_{\left\{\bar{Y}>0\right\}} [\pi_0 + (1-\pi_0) F_1]
\end{array}$$

and the corresponding PDF is given by

$$\begin{array}{lcllcl}
f_0 & := & f_{Yy}(\bar{Y}, \bar{y}|t, \bar{Y}=0) & = & f_y(\bar{y}|t)\pi_0 \\
f_1 & := & f_{Yy}(\bar{Y}, \bar{y}|t, \bar{Y}>0) & = & g_Y(\bar{Y}) \cdot f_y(\bar{y}|t) \cdot \mathrm{cop}[G_Y(\bar{Y}), S_y\left(\bar{y}|t\right)] \\
f_{Yy} & := & f_{Yy}(\bar{Y}, \bar{y}|t, \bar{Y}\geq 0) & = & \mathbf{1}_{\left\{\bar{Y}=0\right\}} f_0 + \mathbf{1}_{\left\{\bar{Y}>0\right\}} f_1
\end{array}$$

with copula function derivative

$$\mathrm{cop}(u,v) = \frac{\delta^2 \mathrm{Cop}(u,v)}{\delta u \delta v}$$

3.4 Model specification

For the exit TIMING regression, we apply Cox (1972)'s multiplicative hazard modeling idea to specify a parametric Weibull survival function. Our approach allows the integration of exogenous, time-variant variables into the exit TIMING regression, which is in contrast to the analyses of Giot & Schwienbacher (2007), Félix et al. (2014), and Jenkinson & Sousa (2015) focusing on internal, time-invariant covariates to examine empirical exit routes in VC and BO. The survival model construction and associated likelihood function are described in 3.5.

According to our theoretical framework, just one single distribution cash flow D_i and one single contribution cash flow C_i per asset is scheduled. However, in real datasets, multiple investment and divestment cash flows can be observed for a given company. To account for this practical consideration, we redefine the MULTIPLE regression's dependent variable as

$$Y_{i,t} = \frac{\check{D}_i(t)}{V_i(t) + \check{C}_i(t)} \geq 0 \tag{1}$$

Here $\check{C}_i(t)$ resp. $\check{D}_i(t)$ represents the sum of all contribution resp. distribution cash flows occurring after t:

$$\check{C}_i(t) := \sum_{\tau} C_{i,\tau} \mathbf{1}_{\{t<\tau\}} \qquad \text{and} \qquad \check{D}_i(t) := \sum_{\tau} D_{i,\tau} \mathbf{1}_{\{t<\tau\}}$$

The corresponding marginal two-part model consists of a logistic regression for π_0 and a generalized linear model relying on a Gamma distribution for G_Y, which are adopted from the R package GAMLSS introduced by Rigby & Stasinopoulos (2005).

To detect further dependency between both marginal models a 180-degree rotated Joe copula is tested

$$\mathrm{Cop}_{\mathrm{Joe}}(u, v; \theta) \quad = \quad 1 - [u^\theta + v^\theta - u^\theta v^\theta]^{1/\theta}$$

with parameter $\theta \geq 1$.

3.5 Exit timing: Parametric multiplicative hazard rate model

The survival function describing the exit timing of private equity fund investments is given by $S(t) = \mathbb{P}[T > t \,|\mathscr{F}_t]$ with total holding period $T = d - c$, entry date c, and exit date d. Let the point process $N_{i,j}^{(d)}(t) = \mathbf{1}_{\{d_i \leq t\}}$ (with indicator function $\mathbf{1}_{\{\}}$) model the exit TIMING $T_{i,j}$ of the ith company belonging to the jth fund in line with the intensity based definition of Bremaud (1981, II.3)

$$\mathbb{E}\left[\int_0^\infty dN_{i,j}^{(d)}(u) \middle| \mathscr{F}_t\right] = \mathbb{E}\left[\int_0^\infty h_{i,j}\left(u \middle| \mathbf{X}(u)\right) du \middle| \mathscr{F}_t\right]$$

The associated random intensity process is characterized according to Andersen & Gill (1982)

$$h_{i,j}\left(t \,|\mathbf{X}(t)\right) = h_0(t)\, Z_{i,j}^{(\mathrm{cens})}(t) \exp\left(\beta^\top \mathbf{X}(t)\right)$$

with $\mathscr{F}_t$-measurable censoring process

$$Z_{i,j}^{(\mathrm{cens})}(t) = \mathbf{1}_{\{t\, >\, c_{i,j}\}} \mathbf{1}_{\{t\, \leq\, d_{i,j}\}}$$

and $\mathscr{F}_t$-measurable covariate process $\mathbf{X}(t)$ introduced in section 3.1. Further, we assume Weibull distributed exit TIMINGS, i.e., a parametric model for the base hazard function $h_0(t)$. The corresponding Weibull cumulative hazard function is given by a simple closed form expression

$$H_0^{(\mathrm{wb})}(t \,|\xi_y) = \int_0^t h_0^{(\mathrm{wb})}(u \,|\xi_y)\, du = \left(\frac{t}{\mathrm{scale}_{\mathrm{wb}}}\right)^{\mathrm{shape}_{\mathrm{wb}}}$$

with parameter vector $\xi_y = (\mathrm{shape}_{\mathrm{wb}},\, \mathrm{scale}_{\mathrm{wb}})$.

The survival function for a parametric Cox model with time-variant covariates is calculated by integrating over the multiplicative intensity process

$$S_{\text{Cox}}(t) = \exp\left(-\int_0^t h_0\left(u \,|\xi_y\right) Z_{i,j}^{(\text{cens})}(u) \exp\left(\beta^\top \mathbf{X}(u)\right) du\right) \tag{2}$$

To account for the quarterly reporting practice of PEFs, we apply the following full likelihood approach to estimate the parametric Cox model with (i) time-variant covariates, (ii) an interval-censored data structure, and (iii) possible final (i.e., nonrandom) right-censoring

$$L\left(\beta, \xi_y \,|\mathbf{T}, \mathbf{X}\right) = \prod_{j=1}^{J} \left[\prod_{i=1}^{n_j} A^{(1-\mathcal{R}_{i,j})} B^{\mathcal{R}_{i,j}}\right] \tag{3}$$

with interval-censored part

$$A = S_{\text{Cox}}\left(T_{i,j}^{(L)}\right) - S_{\text{Cox}}\left(T_{i,j}^{(R)}\right)$$

and right-censored part

$$B = S_{\text{Cox}}\left(T_{i,j}^{(R)}\right)$$

where the left and right boundaries of the exit Timing interval are given by

$$T_{i,j}^{(L)} = d_{i,j}^{(L)} - c_{i,j}^{(L)} \qquad \text{and} \qquad T_{i,j}^{(R)} = \min\left(f_{i,j}, d_{i,j}^{(R)}\right) - c_{i,j}^{(L)}$$

with quarterly left and right boundaries defined according to

$$c_{i,j}^{(L)} < c_{i,j} < c_{i,j}^{(R)} \qquad \text{and} \qquad d_{i,j}^{(L)} < d_{i,j} < d_{i,j}^{(R)}$$

The final observation time for a given asset is denoted by $f_{i,j}$ with corresponding right censoring indicator

$$\mathcal{R}_{i,j} = \mathbf{1}_{\left\{f_{i,j} < d_{i,j}^{(R)}\right\}}$$

This allows us to construct the adjusted likelihood for non-informative right-censoring (Aalen et al., 2008, section 5.1.2). Here n_j gives the number of investments for the jth fund and $j = 1, 2, \ldots, J$ counts the number of distinct funds in the dataset.

Further, we assume a step function for the dynamic covariate process $\mathbf{X}(t)$ since we have to integrate over the history of this process in the course of calculating the survival function. In the maximum likelihood estimation procedure for the Cox model with a Weibull hazard rate function we compute a quarterly time-discrete approximation of equation (2), i.e.,

$$S_{\text{Cox}}^{(\text{wb})}(t) = \exp\left(-\sum_{q:t_q\leq t} \exp\left(\beta^{\top}\mathbf{X}(t_q)\right)\Delta H_{t_q}\right) \tag{4}$$

where the cumulative hazard function difference replaces the integral

$$\begin{aligned}\Delta H_{t_q} &= \int_{t_{q-1}}^{t_q} Z_{i,j}^{(\text{cens})}(u) h_0^{(\text{wb})}(u\,|\xi_y)\,du \\ &= Z_{i,j}^{(\text{cens})}(t_q)\left[H_0^{(\text{wb})}(t_q\,|\xi_y) - H_0^{(\text{wb})}(t_{q-1}\,|\xi_y)\right]\end{aligned} \tag{5}$$

with quarter start date t_{q-1} and quarter end date t_q. The covariate information $\mathbf{X}(t_q)$ is here assumed to be relevant for the time in between t_{q-1} and t_q.

4 Data & model estimation

4.1 Deal-level dataset

For the empirical application of the joint regression model, we use deal-level data (stemming from a fund-of-fund program[4]) split into a BO and a VC subset. All investments had been entered between 1998-09-30 and 2016-12-31 (cf. table 1). The underlying companies originate from 144 BO and 98 VC funds. Although these 242 PEFs cover only a relatively small portion of the total private equity fund universe, the number of deals in our dataset is more extensive than in the datasets of Schmidt et al. (2010); Guo et al. (2011); Braun et al. (2011); Valkama et al. (2013); Félix et al. (2014). Naturally, some commercial deal-level databases like CEPRES, as used by Buchner (2016), or VentureXpert, as used by Giot & Schwienbacher (2007), contain more deals and cover a more extended period than our dataset.

The empirical distributions of both dependent variables are visualized in figure 2. For both fund types, the maximum holding period is approximately 15 years, and at least 10% of the MULTIPLE observations are precisely zero. As of 2016-12-31, the entire fund-of-fund program exhibits an MSCI World KS-PME ratio of 85%. When compared to all fund-level TVPIs in the Preqin universe, 13% of the underlying funds are top-quartile funds, 18%

[4] The proprietary dataset is provided by AssetMetrix GmbH, a service provider for private capital investors and fund managers.

are in the second quartile, 22% are in the third quartile, and 47% are in the fourth quartile. Both public and private market benchmarks indicate that the returns in our deal sample are likely below average. However, missing some upward potential in MULTIPLE parameter estimation might not be overly problematic when focusing on (downward) risk assessment.

The MULTIPLE regression dataset excludes all non-exited observations, and all companies entered after 2009-12-31 to mitigate possible sample selection bias. Allowing exit observations of recently entered companies to infiltrate the MULTIPLE analysis dataset, causes (upward) biased estimates if there prevails a significant (negative) relation between holding period and exit performance. However, precisely this presumed connection is one of the subjects under investigation in this paper. As conjectured in Gredil (2018), we likewise expect fast deal exits (so-called quick-flips) to come with an extraordinarily good deal performance so that the fund manager receives performance fees (so-called carry). On the other hand, reasonably bad performing deals are held for longer times to harvest (at least) some base management fees. Ignoring these considerations and including also recently entered exits in the MULTIPLE dataset, most likely, leads to overly optimistic risk and return assessments. Finally, the cut-off date 2009-12-31 is chosen as a discretionary compromise between sample selection concerns and data comprehensiveness, since with a dataset starting in the late 1990s, basically all deals must be excluded when solely deals of completely liquidated fund vintages are included in the MULTIPLE regression.

4.2 Explanatory variables

In both marginal regression models, we focus on (i) public market and (ii) timing-related covariates. Public market predictors are especially useful for stress testing in the risk management context. Timing-related covariates are interesting as this paper wants to investigate the connection between exit timing and performance.

The common public market variables, shared by both marginal analyses, cover high yield spreads[5] and public equity performance[6]. The TIMING regression is not capable of incorporating holding period $(t-c)$ and time to exit $(d-t)$ as independent variables since they are directly derived from the

[5]We use the BofA Merrill Lynch US High Yield Option-Adjusted Spread (https://fred.stlouisfed.org/series/BAMLH0A0HYM2).

[6]In the TIMING regression we incorporate monthly Public Equity Returns (i.e., $\frac{\text{Index}_{t+1}}{\text{Index}_t} - 1$) and in the MULTIPLE regression we use same-horizon Public Equity Multiples (i.e., $\frac{\text{Index}_d}{\text{Index}_t}$) both calculated with the MSCI World Total Return Index in US Dollar.

Table 1: Dataset summary. Entries and exits per year for BO and VC dataset.

	Entries		Exits	
Year	BO	VC	BO	VC
1998	15	13	0	0
1999	42	125	2	0
2000	141	457	1	13
2001	72	238	28	77
2002	202	307	63	102
2003	99	277	58	154
2004	150	317	73	135
2005	153	233	84	185
2006	188	211	113	173
2007	149	136	130	179
2008	149	128	69	177
2009	65	114	82	168
2010	134	122	96	179
2011	138	135	96	190
2012	143	99	105	168
2013	107	81	105	175
2014	84	52	130	176
2015	87	19	158	141
2016	73	3	149	144
Observed exits (before 2016-12-31)	-	-	1,542	2,536
Censored exits (after 2016-12-31)	-	-	649	531
Observed exits (entry before 2009-12-31)	-	-	1,231	2,179
Thereof with unique company ID	-	-	1,108	1,836

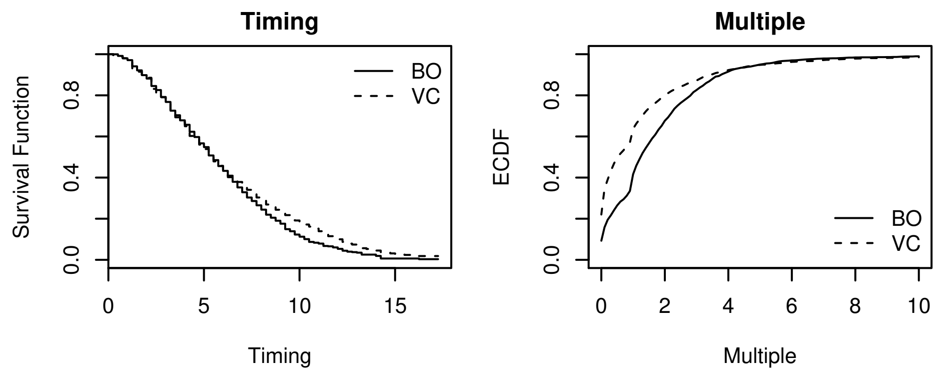

Figure 2: Empirical distribution functions of dependent variables. Kaplan-Meier estimate of the survival function for the TIMING variable in the left plot and empirical cumulative distribution function (ECDF) for the entry-to-exit MULTIPLE variable in the right plot.

dependent variable. Instead, the fund age at entry date serves as the only time-related TIMING covariate.

We just consider one deal-specific predictor variable: the deal-level Residual Value to Paid In (RVPI), defined as value proxy to cumulative contributions ratio (V/C). The RVPI ratio must be regarded as time-variant *internal* variable (Kalbfleisch & Prentice, 2002, section 6.3.2) and therefore is excluded from the set of possible TIMING predictors. Other candidates of deal- or fund-specific covariates can be found in the articles cited in section 2, but are not tested in our rather methodological than empirical paper. Similarly, potential public risk factors for PE investments are summarized by Korteweg (2019).

4.3 Estimation procedure

4.3.1 Timing

The TIMING regression includes all exited and right-censored observations. We explicitly keep multiple entries for a company with numerous fund investors since each fund manager endogenously determines the entry and exit TIMING[7]. Thus, the estimation procedure for the marginal TIMING

[7]The assumption of an endogenous exit decision is just valid for non-syndicated deals. This seems to be true for 81% of our companies with multiple fund investors, as these company investments show distinct entry and/or exit dates.

model, i.e., maximizing equation (3), is relatively straightforward. However, it is computationally more intensive than a parametric multiplicative hazard rate regression with only time-invariant covariates since it involves the numerical integration (stepwise approximation) over the hazard rate function in equation (4). The numerical maximum likelihood optimization for our Cox Weibull model is performed in R by the function optimx(... , method= "nlminb") from the optimx package.

4.3.2 Multiple

The estimation procedure for the marginal MULTIPLE model is more intricate since we have to account for (i) the longitudinal data structure and (ii) economically negligible observations. Unfortunately, standard panel model estimators are not applicable since they commonly use a semi-parametric specification combined with a Heteroskedasticity- and Autocorrelation-Consistent (HAC) estimator for the residuals. To estimate our fully parametric MULTIPLE model in a similar manner, we need to specify an auxiliary parametric model for the within company autocorrelation. However, to avoid numerical convergence problems[8] and misspecification[9], we alternatively propose a simple one-per-company resampling scheme to resolve the MULTIPLE autocorrelation issue. In each step of our iterative procedure, exactly one observation per company identifier is randomly selected to enter the likelihood optimization. Sampling by company identifier also resolves the issue when multiple funds invest in the same company.

Moreover, observations with a deal-level RVPI of less than 10% are excluded within the resampling algorithm since we regard them as economically irrelevant. These small RVPI situations usually arise at the end of deal lifetimes due to (i) distressed deals which got significantly written down and, more importantly, (ii) after big partial sales of almost all company shares, which arguably constitute the economically relevant exit events. We assume that these small RVPI observations (occurring after the economic exit date) add more noise than information, as fund managers seem to come up with sloppy deal appraisals in these situations, i.e., the remaining deal valuations and future cash flows seem not to be related in an economically meaningful way anymore. The exclusion of these cases is just one possible RVPI related weighting method and could be replaced by more elaborate approaches, e.g., weighting the likelihood function by the RVPI in the optimization procedure.

[8]Even a simple AR(1) autocorrelation model always terminated without convergence. R code: gamlss::re(random=∼predictor|ID, correlation=nlme::corARMA(p=1,q=0)).

[9]Economically negligible observations exacerbate correct specification.

The choice of a resampling based estimation method is associated with high computational costs. However, it enables applying an informative yet straightforward marginal MULTIPLE model and simultaneously provides resampling-based standard error estimates (free of charge). The generalized linear models for π_0 and G_Y are separately estimated by the functions gamlss(..., family = BI(mu.link = logit)) and gamlss(..., family = GA)), resp., from the R package GAMLSS (Rigby & Stasinopoulos, 2005).

4.3.3 Copula

For the copula estimation, we use the inference functions for margins approach of Joe & Xu (1996). In this procedure, first, the marginal models are estimated from separately maximized univariate likelihoods. Here, the survival model for TIMING incorporates non-exited investments in contrast to the MULTIPLE model. The second step examines the dependence parameter's significance. The 180-degree rotated Joe copula derivative is obtained by the function BiCopPDF(..., family = 16) from the R package VineCopula.

4.4 Parameter estimates

4.4.1 Timing

The coefficient estimates associated with the multiplicative hazard rate model from appendix 3.5 explain the impact of (i) public equity returns, (ii) high yield spreads, and (iii) the fund age at entry on the exit TIMING of individual fund investments (cf. table 2).

Favorable public market conditions, i.e., high public equity returns and low high yield spreads, and a high fund age at entry date result in faster exit TIMINGS. The corresponding Akaike Information Criterion (AIC) values indicate that the relative quality of TIMING models with covariates is superior to a Weibull distribution model without covariates for both BO and VC datasets, since models with minimum AIC are to be preferred[10]. For the BO subset, the minimum AIC model (a) contains all three covariates, but for the VC subset, the minimum AIC model (b) contains just public equity returns and the fund age at entry as covariates.

In summary, the public market related estimates indicate that high yield spreads posses more predictive power for the BO set, and public equity returns posses more predictive power for the VC set. The fund age at entry

[10] The AIC is calculated as $AIC = 2k - 2\ln\left(\hat{L}\right)$ where k gives the number of parameters used in a given regression and $\hat{L}$ denotes the maximized likelihood value.

Table 2: Parameter and coefficient estimates of the Timing regression. Three distinct sets of covariates (a - c) are tested for each fund type. The standard errors (in parentheses) are obtained from the corresponding Hessian matrix.

Timing estimates	Buyout			Venture Capital		
Variables	(a)	(b)	(c)	(a)	(b)	(c)
Public equity return	1.892	2.478	-	5.201	5.169	-
	(1.084)	(0.959)	-	(0.756)	(0.775)	-
High yield spread	-3.071	-	-3.588	0.651	-	-0.43
	(1.076)	-	(1.051)	(0.674)	-	(0.67)
Fund age (at entry)	0.086	0.088	0.085	0.036	0.036	0.037
	(0.014)	(0.014)	(0.014)	(0.01)	(0.01)	(0.01)
Scale (Weibull)	6.44	7.221	6.261	7.554	7.355	6.977
	(0.308)	(0.19)	(0.285)	(0.269)	(0.165)	(0.24)
Shape (Weibull)	1.651	1.654	1.652	1.474	1.474	1.482
	(0.034)	(0.034)	(0.034)	(0.269)	(0.165)	(0.24)
AIC (including covariates)	12,598	12,604	12,599	21,058	21,057	21,103
AIC (without covariates)	12,647	12,647	12,647	21,113	21,113	21,113

effect is highly significant for both fund types; however, the magnitude of this effect is stronger for the BO set.

4.4.2 Multiple

The coefficient estimates obtained for the two-part model introduced in section 3 explain the impact of (i) public market, (ii) private (proxy) valuation, and (iii) exit-timing-related covariates on the exit Multiple of individual fund investments (cf. table 3).

Favorable public market conditions, i.e., now a high public equity multiple and a low high yield spread, lead to high Multiple estimates in both sub-models since the signs of the public equity multiple coefficients are positive and the signs of the high yield spread coefficients are negative for π_0 and $\mu(G_Y)$. Here π_0 denotes the probability of default, i.e., a zero Multiple, and $\mu(G_Y)$ represents the Gamma distribution mean. Unsurprisingly, low company valuation proxies, when compared to the initial investment amount, increase the probability of default. The time-related covariates are (historical) holding period and (future) time-to-exit. For both BO and VC, longer holding periods decrease the probability of default π_0, but negatively

Table 3: Parameter and coefficient estimates of the two-part MULTIPLE regression. The default probability π_0 is estimated by a logit model, and the Gamma distribution G_Y is specified by parameters μ, σ. The resampling procedure is iterated 1,000 times for each fund type. The estimates are mean and standard deviation (in parentheses) that are naturally estimated within the resampling based methodology.

MULTIPLE estimates	Buyout			Venture Capital		
Covariates	$1-\pi_0$	$\mu(G_Y)$	$\sigma(G_Y)$	$1-\pi_0$	$\mu(G_Y)$	$\sigma(G_Y)$
Intercept	1.551	0.846	-0.083	1.141	0.671	0.378
	(0.144)	(0.080)	(0.034)	(0.106)	(0.108)	(0.020)
Holding period	0.122	-0.048	-0.035	0.134	-0.003	-0.030
	(0.033)	(0.015)	(0.014)	(0.023)	(0.014)	(0.009)
Time to exit	-	-0.041	0.086	-	-0.39	0.051
	-	(0.024)	(0.009)	-	(0.019)	(0.005)
RVPI - 1 (deal-level)	0.818	-	-	0.262	-	-
	(0.186)	-	-	(0.067)	-	-
Public equity multiple - 1	1.218	0.720	-	0.962	0.701	-
	(0.158)	(0.125)	-	(0.102)	(0.167)	-
High yield spread	-4.275	-3.763	-	-6.176	-4.409	-
	(1.927)	(1.012)	-	(1.399)	(1.688)	-
Link function	logit	log	log	logit	log	log
AIC (including covariates)	935	2,775	2,775	1,988	2,409	2,409
AIC (without covariates)	979	2,923	2,923	2,058	2,525	2,525

affect both the non-default MULTIPLE mean $\mu(G_Y)$ and variance-related term $\sigma(G_Y)$. For VC, the holding period effect on $\mu(G_Y)$ is not statistically significant. For both BO and VC, high future time-to-exits decrease $\mu(G_Y)$ but increase $\sigma(G_Y)$.

As we use log-link functions for μ and σ, $\mu(G_Y) = \exp(\beta_\mu^\top \mathbf{X})$ and $\sigma(G_Y) = \exp(\beta_\sigma^\top \mathbf{X})$ where β_μ, β_σ denote the coefficient estimates displayed in table 3. The variance of the continuous Gamma distribution that models the non-default MULTIPLE $Y^+ > 0$ is defined as $\mathrm{Var}(Y^+) = \sigma(G_Y)^2 \cdot \mu(G_Y)^2$. The default probability is calculated in the logit model as $\pi_0 = 1 - o(1+o)^{-1}$ with $o = \exp(\beta_\pi^\top \mathbf{X})$.

The AIC values for the zero and continuous part of the MULTIPLE model indicate that the relative qualities of the full covariate regressions are superior to their corresponding intercept only equivalents for both BO and VC datasets.

4.4.3 Copula

The 180-degree rotated Joe copula parameter estimates are 1.135 (0.018) for the BO set and 1.101 (0.015) for the VC set (with resampling based standard errors in parentheses). As a parameter estimate of $\theta = 1$ implies conditional independence between the marginal MULTIPLE and TIMING models, the estimates suggest that we require a small copula-induced correction of their collective behavior (for both BO and VC).

A statistical significant parameter estimate $\theta > 1$ for the 180-degree rotated Joe copula lets our joint model generate more (extremely) small MULTIPLES for (extremely) large TIMING realizations as compared to a bivariate uniform distribution. As a result, our copula model displays the propensity to produce very long holding periods for bad performing deals, as outlined in Gredil (2018). Therefore, MULTIPLE estimates for shorter holding periods will be larger than in the conditionally independent case.

5 Monte Carlo model and discussions

5.1 Portfolio aggregation

Our exit dynamics model is suited for straightforward and computationally inexpensive cash flow simulations, which help to genuinely understand the risk of a given static PEF portfolio since the underlying framework can process detailed asset-level information. Here bottom-up portfolio aggregation relies on conditional independence, i.e., the dependence between portfolio companies is solely introduced by common covariates $\mathbf{X}$. However, due to the time-variant independent variables used within the TIMING model, our Monte Carlo simulation can capture the exit clustering effect observed in times with excellent public market conditions (so-called hot exit markets). In our reduced form model, a period with strong [weak] public equity market performance increases [decreases] the probability of an exit in that period for each deal held in the PEF portfolio. So we explicitly avoid a (self-exciting point process like) modeling approach where observed deal exits trigger further deal exits in the short term. The self-exciting access may be even structurally wrong since exit clustering is probably best explained by favorable public market conditions that are common for all deals[11].

[11] Many authors find public market covariates that affect the exit decision and timing: Giot & Schwienbacher (2007) use variable IPOACTIVITY. Jenkinson & Sousa (2015) include the macroeconomic variables (i) local stock index return, (ii) capital commitment index return, and (iii) Fed tightening index in their exit decision model; in their Cox model,

5.2 Cash-Flow-at-Risk simulation

Liquidity and cash flow risk at the α-level for a given fixed horizon z can be conveniently assessed by a portfolio-level Cash-Flow-at-Risk (CFaR) measure that relies on Monte Carlo simulation results

$$\mathrm{CFaR}_{\alpha,z}(Y^{(\mathrm{PF})}) = \inf\left\{x \geq 0 : F_{Y^{(\mathrm{PF})}}(x,z) > \alpha\right\}$$

where the z-horizon specific CDF for the weighted portfolio multiple is given by

$$F_{Y^{(\mathrm{PF})}}(x,z) = \mathbb{P}\left[x \geq \sum_{i=1}^{n} w_{it} Y_{it} \mathbf{1}_{\{y_{it} \leq z\}} \middle| \mathscr{F}_t\right] \tag{6}$$

with portfolio weights w_{it}. Monte Carlo simulation formally constitutes the application of the MULTIPLE conditional on TIMING model specified in section 3 with the parameter estimates from section 4 for each portfolio company. Specifically, our simulation procedure relies on inverse transform sampling, i.e.,

$$\begin{aligned} \widetilde{Y} &= G_Y^{-1}\left(\widetilde{U}_{G_Y} \middle| \xi_Y, \widetilde{\mathbf{X}}, \widetilde{y}\right) \mathbf{1}_{\left\{\widetilde{U}_\pi > \pi_0(\widetilde{\mathbf{X}})\right\}} + 0 \\ \widetilde{y} &= S_y^{-1}\left(S_y(t) \cdot \widetilde{U}_y \middle| \xi_y, \widetilde{\mathbf{X}}\right) \end{aligned}$$

where the tilde symbolizes the simulated nature of a given random variable. Possible future covariate process paths $\widetilde{\mathbf{X}}$ can be held fixed or re-simulated in each iteration step. The default random variable $\widetilde{U}_\pi$ is distributed uniformly i.i.d. and the bivariate uniforms $\left(\widetilde{U}_y, \widetilde{U}_{G_Y}\right)$ are generated from the 180-degree rotated Joe copula $\mathrm{Cop}_{\mathrm{Joe}}$.

they include country and exit year fixed effects. Cumming (2008) incorporates the MSCI return prior to the exit. Schmidt et al. (2010) use multiple measures of hot exit markets.

5.3 Potential benefits of deal-level risk models

Risk regulation frameworks like Solvency II and Basel III privilege look-through approaches that perform the risk assessment on the most granular level. Hence, it is an interesting question if the risk evaluation of private equity fund portfolios always benefits from applying the more detailed deal-level models compared to fund-level models. Here, our two implicit assumptions are that (i) deal-level information is potentially useful (neglection is potentially harmful) and (ii) deal-level information is best processed by deal-level (not by fund-level) models.

From this perspective, deal-level models conceivably offer benefits for different applications. The modeling of future fund-level fees is best studied on deal level to emulate complex carry arrangements. The analysis of portfolio-level implications associated with particular industry and company stage allocations customarily requires deal-level knowledge. Especially for small portfolios, deal-level models reveal diversification effects that are hardly grasped by fund-level models.

$CFaR_{\alpha,z=\infty}$	3 deals, age 2	3 deals, age 8	15 deals, age 2	15 deals, age 8
$\alpha = 0.5\%$	0.0000	0.0061	0.2082	0.3737
$\alpha = 1\%$	0.0005	0.0134	0.2585	0.4410
$\alpha = 5\%$	0.0368	0.1073	0.4512	0.6543
$\alpha = 50\%$	0.9498	1.1130	1.3299	1.4815

Table 4: Monte Carlo simulation example. Cash-Flow-at-Risks with an infinite horizon (i.e., quantile of final fund multiple) for our VC fund example. VC funds with more deals and longer realized holding periods (simply called age) are considered safer by our model.

Figure 3 and table 4 show a simplified Monte Carlo example designed to emphasize the benefit of incorporating deal-level information in the risk management context. Here, the cash flow profiles of four otherwise equal VC funds with number of companies 3 and 15, respectively, and with deal ages (i.e., holding period until now) 2 and 8, respectively, are compared[12]. The study uses historical public market observations up to 2016-12-31; afterward, *one* possible future path is generated by sampling random permutations from the empirical data. Thus, all simulation iterations rely on the same public market scenario. Further, to focus on the deal ages and the number of companies in the respective fund, we set all deal-level RVPIs (V/C) to one

[12] As a technicality, for the funds with deal age equal to 2, one deal age is always set to 8 to justify equal vintage years for all four funds.

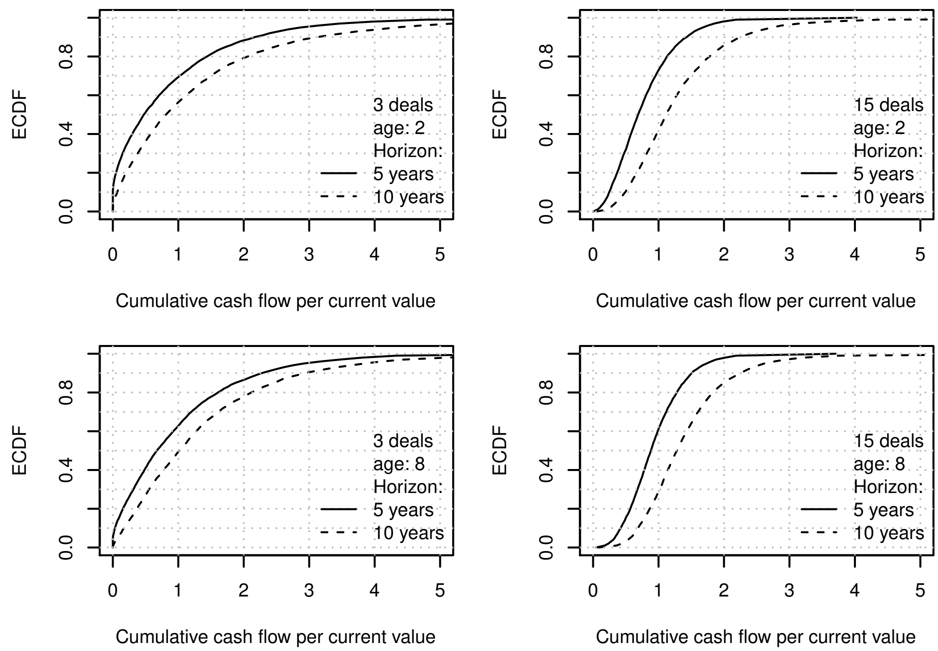

Figure 3: Monte Carlo simulation example. The example compares the cash flows associated with four generic VC funds. The empirical CDF equivalents of equation (6) illustrate varying cash flow risk profiles when the number of fund holdings shifts from 3 (in the left charts) to 15 (in the right charts) and when deal ages shift from 2 (in the top charts) to 8 (in the bottom charts).

and assume equal investment amounts per deal. Each run involves 5,000 iterations.

In figure 3 and table 4, our deal-level model provides considerably distinct exit cash flow profiles when the level of diversification or the deal ages change. The cumulative distribution functions differ, especially in the lower tails, which are particularly important for downside risk management. Typical fund-level models usually neglect the fund characteristics 'number of underlying deals' and 'average deal age' and, thus, often cannot distinguish funds from the same vintage year at all. Hence, risk management for small PEF portfolios can substantially benefit from applying our deal-level model instead of crude fund-level approaches that ignore deal-level information.

Deal-level models are particularly useful for static risk management, i.e., if we just want to assess the riskiness of the current (not future) portfolio composition. The delicate question of how future capital calls are financed can be conveniently disregarded in this static view that only considers the investments presently held by the fund. Bongaerts & Charlier (2009) also adopt this static view in their structural deal-level model for the regulatory capital requirement under Basel II. When the investment cash flows associated with new entries are of interest, models for the deal-by-deal entry dynamics are a natural complement to our deal-level exit model. Here, the discretized (deal-by-deal) modeling of entries again offers a higher granularity than typical fund-level approaches like (Buchner et al., 2010, Paragraph: Modeling Capital Drawdowns).

To summarize, a fair comparison between deal-level and fund-level cash flow models is just feasible if (i) the deal-level model accounts for future investment cash flows and (ii) the fund-level model accounts for available deal-level information. Table 5 outlines situations where the application of deal-level models seems especially suitable. On the other hand, there are, of course, set-ups that favor fund-level models like, e.g., Buchner (2017).

6 Conclusion

From a methodological viewpoint, this paper takes a new look at the cash flow stream that will be realized over the next years from the $4.99 trillion assets held by private capital funds (as of September 2019; Source: Preqin). More precisely, the divestment behavior of private equity fund deals is studied by a *joint* copula model that formulates the exit MULTIPLE of a given fund investment conditionally on its exit TIMING. Here, the asset-level granularity enables the inclusion of many insightful covariates that are

Table 5: Deal- vs. fund-level model comparison. Appropriateness of our deal-level model and the fund-level model of Buchner (2017) in the risk management context.

	Our deal exit model	Buchner (2017)
level	deal-level	fund-level
cash flows modeled	exit cash flows	con- & distributions
reasonable extension	entry cash flows	deal information
public covariates	market return, high yield spread	market return
Appropriate if:		
portfolio size	small	big
portfolio composition	heterogeneous	homogeneous
deal idiosyncrasy	high	low
many co-investments	yes	no
industry matters	yes	no
future deal entries matter	no	yes
future commitments matter	no	yes
timing of future entries	known	unknown
modeling of fees	explicit	implicit

otherwise infeasible in pure fund-level approaches due to identifiability issues. In our empirical analysis, public market and time-related predictors significantly affect both the deal-level exit TIMING and MULTIPLE of BO and VC investments. Realistic cash flow scenarios for a given static PEF portfolio (ignoring future fund investments) can be ultimately obtained by Monte Carlo simulations that draw on these model estimates.

In our view, sophisticated cash flow projections are a vital tool to improve the risk understanding of PEF vehicles since yet private equity fund stakes cannot be traded on liquid secondary markets. Further, the intrinsic risk of undiversified PEFs (with just a few company holdings) may be commonly underestimated by models with implicit or explicit diversification assumptions. On the other hand, our deal-level approach is capable of reproducing the high default probabilities empirically observed for single BO or VC deals. Naive fund investors may benefit from this improved risk perception, while confident fund managers may hardly consent to these estimates in their subjective risk assessments.

Acknowledgements

We thank Markus Rieder, Kevin Gruber, Marcus Pietz, Sarah Musiol, Holger Fink, Stefan Mittnik, Michael Rockinger, and the participants in the CEQURA Junior Research Workshop 2017 and CEQURA Risk Conference 2018. We thank the anonymous referees for very constructive comments and suggestions. Especially we thank AssetMetrix GmbH for the generous access to data. The views expressed in this paper are those of the authors and do not necessarily reflect those of AssetMetrix GmbH.

Declaration of interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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